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SECTION 2

ELECTRIC AND MAGNETIC CIRCUITS*

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2.1 ELECTRIC AND MAGNETIC CIRCUITS

Definition of Electric Circuit. An *electric circuit* is a collection of electrical devices and components connected together for the purpose of processing information or energy in electrical form. An electric circuit may be described mathematically by ordinary differential equations, which may be linear or

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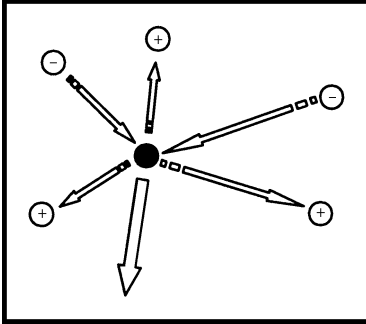


FIGURE 2-1 Electric charges.

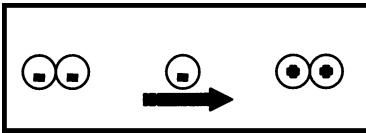


FIGURE 2-2 Electric voltage.

nonlinear, and which may or may not be time varying. The practical effect of this restriction is that the physical dimensions are small compared to the wavelength of electrical signals. Many devices and systems use circuits in their design.

Electric Charge. In circuit theory, we postulate the existence of an indivisible unit of charge. There are two kinds of charge, called *negative* and *positive* charge. The negatively charged particle is called an *electron*. Positive charges may be atoms that have lost electrons, called *ions*; in crystalline structures, electron deficiencies, called *holes*, act as positively charged particles. See Fig. 2-1 for an illustration. In the International System of Units (SI), the unit of charge is the coulomb (C). The charge on one electron is 1.60219×10^{-19} C.

Electric Current. The flow or motion of charged particles is called an *electric current*. In SI units, one of the fundamental units is the ampere (A). The definition is such that a charge flow rate of 1 A is equivalent to 1 C/s. By convention, we speak of current as the flow of positive charges. See Fig. 2-2 for an illustration. When it is necessary to consider the flow of negative charges, we use appropriate modifiers. In an electric circuit, it is necessary to control the path of current flow so that the device operates as intended.

Voltage. The motion of charged particles either requires the expenditure of energy or is accompanied by the release of energy. The voltage, at a point in space, is defined as the work per unit charge (joules/coulomb) required to move a charge from a point of zero voltage to the point in question.

Magnetic and Dielectric Circuits. Magnetic and electric fields may be controlled by suitable arrangements of appropriate materials. Magnetic examples include the magnetic fields of motors, generators, and tape recorders. Dielectric examples include certain types of microphones. The fields themselves are called *fluxes* or *flux fields*. Magnetic fields are developed by magnetomotive forces. Electric fields are developed by voltages (also called *electromotive forces*, a term that is now less common). As with electric circuits, the dimensions for dielectric and magnetic circuits are small compared to a wavelength. In practice, the circuits are frequently nonlinear. It is also desired to confine the magnetic or electric flux to a prescribed path.

2.1.1 Development of Voltage and Current

Sources of Voltage or Electric Potential Difference. A voltage is caused by the separation of opposite electric charges and represents the work per unit charge (joules/coulomb) required to move the charges from one point to the other. This separation may be forced by physical motion, or it may be initiated or complemented by thermal, chemical, magnetic, or radiation causes. A convenient classification of these causes is as follows:

- a. Friction between dissimilar substances
- b. Contact of dissimilar substances
- c. Thermoelectric action
- d. Hall effect
- e. Electromagnetic induction
- f. Photoelectric effect
- g. Chemical action

Voltage Effect or Contact Potential. When pieces of various materials are brought into contact, a voltage is developed between them. If the materials are zinc and copper, zinc becomes charged positively and copper negatively. According to the electron theory, different substances possess different tendencies to give up their negatively charged particles. Zinc gives them up easily, and thus, a number of negatively charged particles pass from it to copper. Measurable voltages are observed even between two pieces of the same substance having different structures, for example, between pieces of cast copper and electrolytic copper.

Thomson Effect. A temperature gradient in a metallic conductor is accompanied by a small voltage gradient whose magnitude and direction depend on the particular metal. When an electric current flows, there is an evolution or absorption of heat due to the presence of the thermoelectric gradient, with the net result that the heat evolved in a volume interval bounded by different temperatures is slightly greater or less than that accounted for by the resistance of the conductor. In copper, the evolution of heat is greater when the current flows from hot to cold parts, and less when the current flows from cold to hot. In iron, the effect is the reverse. Discovery of this phenomenon in 1854 is credited to Sir William Thomson (Lord Kelvin), an English physicist.

The Thomson effect is defined by

$$q = \rho J^2 - \mu J \frac{dT}{dx}$$

where q is the heat production per unit volume, ρ is the resistivity of the material, J is the current density, μ is the Thomson coefficient, and dx/dT is the temperature gradient.

Peltier Effect. When a current is passed across the junction between two different metals, an evolution or an absorption of heat takes place. This effect is different from the evolution of heat described by ohmic (i^2r) losses. This effect is reversible, heat being evolved when current passes one way across the junction, and absorbed when the current passes in the opposite direction. The junction is the source of a Peltier voltage. When current is forced across the junction against the direction of the voltage, a heating action occurs. If the current is forced in the direction of the Peltier voltage, the junction is cooled. Refrigerators are constructed using this principle. Since the Joule effect (see Sec. 2.1.8) produces heat in the conductors leading to the junction, the Peltier cooling must be greater than the Joule effect in that region for refrigeration to be successful. This phenomenon was discovered by Jean Peltier, a French physicist, in 1834.

The Peltier effect is defined by

$$Q = \Pi_{AB} \cdot I$$

where Q is the heat absorption per unit time, $Q = \Pi_{AB}$ is the Peltier coefficient, and I is the current.

Seebeck Effect. When a closed electric circuit is made from two different metals, two (or more) junctions will be present. If these junctions are maintained at different temperatures, within certain ranges, an electric current flows. If the metals are iron and copper, and if one junction is kept in ice while the other is kept in boiling water, current passes from copper to iron across the hot junction. The resulting device is called a *thermocouple*, and these devices find wide application in temperature measurement systems. This phenomenon was discovered in 1821 by Thomas Johann Seebeck.

The Seebeck effect is defined by

$$V = \int_{T_1}^{T_2} S_B(T) - S_A(T) dT$$

where V is the voltage created, S is the Seebeck coefficient, and T is the temperature at the junction.

The Thomson, Peltier, and Seebeck equations are related by

$$\Pi = S \cdot T$$

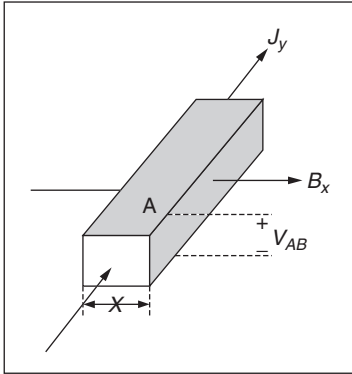


FIGURE 2-3 Hall-effect model.

Hall Effect. When a conductor carrying a current is inserted into a magnetic field that is perpendicular to the field, a force is exerted on the charged particles that constitute the current. The result is that the particles will be forced to the side of the conductor, leading to a buildup of positive charge on one side and negative charge on the other. This appears as a voltage across the conductor, given by

$$V_{AB} = -\frac{J_y B_x x}{en} = -v B_x x \quad (2-1)$$

where x is width of the conductor, B_x is magnetic field strength, J_y is current density, n is charge density, e is electronic charge, and v is velocity of charge flow.

This phenomenon is useful in the measurement of magnetic fields and in the determination of properties and characteristics of semiconductors, where the voltages are much larger than in conductors. See Fig. 2-3. This effect was discovered in 1879.

Faraday's Law of Induction. According to Faraday's law, in any closed linear path in space, when the magnetic flux ϕ (see Sec. 2.1.2) surrounded by the path varies with time, a voltage is induced around the path equal to the negative rate of change of the flux in webers per second.

$$V = -\frac{\partial \phi}{\partial t} \quad \text{volts} \quad (2-2)$$

The minus sign denotes that the direction of the induced voltage is such as to produce a current opposing the flux. If the flux is changing at a constant rate, the voltage is numerically equal to the increase or decrease in webers in 1 s.

The closed linear path (or circuit) is the boundary of a surface and is a geometric line having length but infinitesimal thickness and not having branches in parallel. It can vary in shape or position.

If a loop of wire of negligible cross section occupies the same place and has the same motion as the path just considered, the voltage v will tend to drive a current of electricity around the wire, and this voltage can be measured by a galvanometer or voltmeter connected in the loop of wire. As with the path, the loop of wire is not to have branches in parallel; if it has, the problem of calculating the voltage shown by an instrument is more complicated and involves the resistances of the branches.

For accurate results, the simple Eq. (2-2) cannot be applied to metallic circuits having finite cross section. In some cases, the finite conductor can be considered as being divided into a large number of filaments connected in parallel, each having its own induced voltage and its own resistance. In other cases, such as the common ones of D.C. generators and motors and homopolar generators, where there are sliding and moving contacts between conductors of finite cross section, the induced voltage between neighboring points is to be calculated for various parts of the conductors. These can then be summed up or integrated. For methods of computing the induced voltage between two points, see text on electromagnetic theory.

In cases such as a D.C. machine or a homopolar generator, there may at all times be a conducting path for current to flow, and this may be called a *circuit*, but it is not a closed linear circuit without parallel branches and of infinitesimal cross section, and therefore, Eq. (2-2) does not strictly apply to such a circuit in its entirety, even though, approximately correct numerical results can sometimes be obtained.

If such a practical circuit or current path is made to enclose more magnetic flux by a process of connecting one parallel branch conductor in place of another, then such a change in enclosed flux does not correspond to a voltage according to Eq. (2-2). Although it is possible in some cases to describe a loop of wire having infinitesimal cross section and sliding contacts for which Eq. (2-2) gives correct numerical results, the equation is not reliable, without qualification, for cases of finite cross section and sliding contacts. It is advisable not to use equations involving $\partial \phi / \partial t$ directly on complete circuits where there are sliding or moving contacts.

Where there are no sliding or moving contacts, if a coil has N turns of wire in series closely wound together so that the cross section of the coil is negligible compared with the area enclosed by the coil, or if the flux is so confined within an iron core that it is enclosed by all N turns alike, the voltage induced in the coil is

$$V = -N \frac{\partial \phi}{\partial t} \quad \text{volts} \quad (2-3)$$

In such a case, $N\phi$ is called the *number of interlinkages of lines of magnetic flux with the coil*, or simply, the *flux linkage*.

For the preceding equations, the change in flux may be due to relative motion between the coil and the magnetomotive force (mmf, the agent producing the flux), as in a rotating-field generator; it may be due to change in the reluctance of the magnetic circuit, as in an inductor-type alternator or microphone, variations in the primary current producing the flux, as in a transformer, variations in the current in the secondary coil itself, or due to change in shape or orientation of the loop of coil. For further study, refer to the Web site <http://www.lectureonline.cl.msu.edu/~mmp/applist/induct/faraday.htm>.

2.1.2 Magnetic Fields

Early Concepts of Magnetic Poles. Substances now called *magnetic*, such as iron, were observed centuries ago as exhibiting forces on one another. From this beginning, the concept of magnetic poles evolved, and a quantitative theory built on the concept of these poles, or small regions of magnetic influence, was developed. André-Marie Ampère observed forces of a similar nature between conductors carrying currents. Further developments have shown that all theories of magnetic materials can be developed and explained through the magnetic effects produced by electric charge motions. Magnetic fields may be seen in Fig. 2-4.

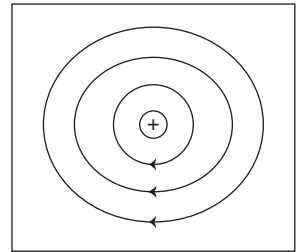


FIGURE 2-4 Magnetic fields.

Ampère's Formula. The magnetic field intensity dB produced at a point A by an element of a conductor ds (in meters) through which there is a current of i A is

$$dH = ids \left(\frac{\sin \alpha}{4\pi r^2} \right) \quad \text{A/m} \quad (2-4)$$

where r is the distance between the element ds and the point A , in meters, and α is the angle between the directions of ds and r . The intensity dH is perpendicular to the plane containing ds and r , and its direction is determined by the right-handed-screw rule given in Fig. 2-45.

The magnetic lines of force due to ds are concentric circles about the straight line in which ds lies. The field intensity produced at A by a closed circuit is obtained by integrating the expression for dH over the whole circuit.

An Indefinitely Long, Straight Conductor. The magnetic field due to an indefinitely long, straight conductor carrying a current of i A consists of concentric circles which lie in planes perpendicular to the axis of the conductor and have their centers on this axis. The magnetic field intensity at a distance of r m from the axis of the conductor is

$$H = \frac{i}{2\pi r} \quad \text{A/m} \quad (2-5)$$

its direction being determined by the right-handed-screw rule (Sec. 2.1.22). See Fig. 2-5 for an illustration.

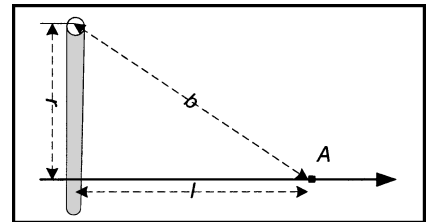


FIGURE 2-5 Magnetic field along the axis of a circular conductor.

Magnetic Field in Air Due to a Closed Circular Conductor. If the conductor carrying a current of i A is bent in the form of a ring of radius r m (Fig. 2-5), the magnetic field intensity at a point along the axis at a distance b m from the ring is

$$H = \frac{r^2 i}{2b^3} = \frac{r^2 i}{2(r^2 + l^2)^{3/2}} \quad \text{A/m} \quad (2-6)$$

When $l = 0$,

$$H = \frac{i}{2r} \quad (2-7)$$

and when l is very great in comparison with r ,

$$H = \frac{r^2 i}{2l^3} \quad (2-8)$$

Within a Solenoid. The magnetic field intensity within a solenoid made in the form of a *torus ring*, and also in the middle part of a *long, straight solenoid*, is approximately

$$H = n_1 i \quad \text{A/m} \quad (2-9)$$

where i is the current in amperes and n_1 is the number of turns per meter length.

Magnetic Flux Density. The magnetic flux density resulting in free space, or in substances not possessing magnetic behaviors differing from those in free space, is

$$B = \mu H = 4\pi \times 10^{-7} H \quad (2-10)$$

where B is in teslas (or webers per square meter), H is in amperes per meter, and the constant $\mu_0 = 4\pi \times 10^{-7}$ is the *permeability* of free space and has units of henrys per meter. In the so-called practical system of units, the flux density is frequently expressed in *lines* or *maxwell per square inch*. The maxwell per square centimeter is called the *gauss*.

For substances such as iron and other materials possessing magnetic density effects greater than those of free space, a term μ_r is added to the relationship as

$$B = 4\pi \times 10^{-7} \mu_r H \quad (2-11)$$

where μ_r is the relative permeability of that substance under the conditions existing in it compared with that which would result in free space under the same magnetic-field-intensity condition. μ_r is a dimensionless quantity.

Magnetic Flux. The magnetic flux in any cross section of magnetic field is

$$\phi = \int B \cos \alpha dA \quad \text{webers} \quad (2-12)$$

where α is the angle between the direction of the magnetic flux density B and the normal at each point to the surface over which A is measured. In the so-called practical system of units, the magnetic *line* (or *maxwell*) is frequently used, where 1 Wb is equivalent to 103 lines.

Density of Magnetic Energy. The magnetic energy stored per cubic meter of a magnetic field in free space is

$$\frac{dW}{dv} = \frac{1}{2} \mu_0 H^2 = 2\pi \times 10^{-7} H^2 = \frac{B^2}{2\mu_0} = \frac{B^2}{8\pi \times 10^{-7}} \quad \text{J/m}^3 \quad (2-13)$$

In magnetic materials, the energy density stored in a magnetic field as a result of a change from a condition of flux density B_1 to that of B_2 can be expressed as

$$dW/dt = \int_{B_1}^{B_2} H dB \quad (2-14)$$

Flux Plotting. Flux plotting by a graphic process is useful for determining the properties of magnetic and other fields in air. The field of flux required is usually uniform along one dimension, and a cross section of it is drawn. The field is usually required between two essentially equal magnetic potential lines such as two iron surfaces. The field map consists of lines of force and equipotential lines which must intersect at right angles. For the graphic method, a field map of curvilinear squares is recommended when the problem is two dimensional. The squares are of different sizes, but the number of lines of force crossing every square is the same.

In sketching the field map, first draw those lines which can be drawn by symmetry. If parts of the two equipotential lines are straight and parallel to each other, the field map in the space between them will consist of lines which are practically straight, parallel, and equidistant. These can be drawn in. Then extend the series of curvilinear squares into other parts of the field, making sure, first, that all the angles are right angles and, second, that in each square the two diameters are equal, except in regions where the squares are evidently distorted, as near sharp corners of iron or regions occupied by current-carrying conductors. The diameters of a curvilinear square may be taken to be the distances between midpoints of opposite sides. An example of flux plotting may be seen in Fig. 2-6.

The magnetic field map near an iron corner is drawn as if the iron had a small fillet, that is, a line issues from an angle of 90° at 45° to the surface.

Inside a conductor which carries current, the magnetic field map is not made up of curvilinear squares, as in free space or air. In such cases, special rules for the spacing of the lines must be used. The equipotential lines converge to a point called the *kernel*.

Computer-based methods are now commonly available to do the detailed work, but the principles are unchanged.

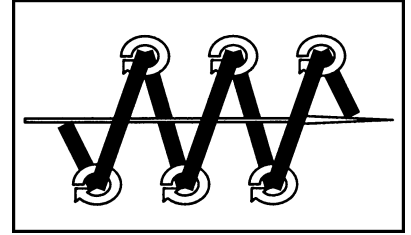


FIGURE 2-6 Magnetic field.

2.1.3 Force Acting on Conductors

Force on a Conductor Carrying a Current in a Magnetic Field. Let a conductor of length l m carrying a current of i A be placed in a magnetic field, the density of which is B in teslas. The force tending to move the conductor across the field is

$$F = Bli \quad \text{newtons} \quad (2-15)$$

This formula presupposes that the direction of the axis of the conductor is at right angle to the direction of the field. If the directions of i and B form an angle α , the expression must be multiplied by $\sin \alpha$.

The force F is perpendicular to both i and B , and its direction is determined by the right-handed-screw rule. The effect of the magnetic field produced by the conductor itself is increase in the original flux density B on one side of the conductor and decrease on the other side. The conductor tends to move away from the denser field. A closed metallic circuit carrying current tends to move so as to enclose the greatest possible number of lines of magnetic force.

Force between Two Long, Straight Lines of Current. The force on a unit length of either of two long, straight, parallel conductors carrying currents of medium (that is, not near masses of iron) is

$$\frac{F}{L} = \frac{2 \times 10^{-7} i_1 i_2}{b} \quad (2-16)$$

where F is in newtons and L (length of the long wires) and b (the spacing between them) are in the same units, such as meters.

The force is an attraction or a repulsion according to whether the two currents are flowing in the same or in opposite directions. If the currents are alternating, the force is pulsating. If i_1 and i_2 are effective values, as measured by A.C. ammeters, the maximum momentary value of the force may be as much as 100% greater than given by Eq. (2-16). The natural frequency (resonance) of mechanical vibration of the conductors may add still further to the maximum force, so a factor of safety should be used in connection with Eq. (2-16) for calculating stresses on bus bars.

If the conductors are straps, as is usual in bus bars, the following form of equation results for thin straps placed parallel to each other, b m apart:

$$\frac{F}{L} = \frac{2 \times 10^{-7} i_1 i_2}{s^2} \left(2s \tan^{-1} \frac{s}{b} - b \log_e \frac{s^2 + b^2}{b^2} \right) \quad \text{N/m} \quad (2-17)$$

where s is the dimension of the strap width in meters, and the thickness of the straps placed side by side is presumed small with respect to the distance b between them.

Pinch Effect. Mechanical force exerted between the magnetic flux and a current-carrying conductor is also present within the conductor itself and is called *pinch effect*. The force between the infinitesimal filaments of the conductor is an attraction, so a current in a conductor tends to contract the conductor. This effect is of importance in some types of electric furnaces where it limits the current that can be carried by a molten conductor. This stress also tends to elongate a liquid conductor.

2.1.4 Components, Properties, and Materials

Conductors, Semiconductors, and Insulators. An important property of a material used in electric circuits is its conductivity, which is a measure of its ability to conduct electricity. The definition of conductivity is

$$\sigma = J/E \quad (2-18)$$

where J is current density, A/m², and E is electric field intensity, V/m.

The units of conductivity are thus the reciprocal of ohm-meter or siemens/meter. Typical values of conductivity for good conductors are 1000 to 6000 S/m. The reciprocal of conductivity is called *resistivity*. Section 4 gives extensive tabulations of the actual values for many different materials. Copper and aluminum are the materials usually used for distribution of electric energy and information. Semiconductors are a class of materials whose conductivity is in the range of 1 mS/m, though this number varies by orders of magnitude up and down. Semiconductors are produced by careful and precise modifications of pure crystals of germanium, silicon, gallium arsenide, and other materials. They form the basic building block for semiconductor diodes, transistors, silicon-controlled rectifiers, and integrated circuits. See Sec. 4.

Insulators (more accurately, dielectrics) are materials whose primary electrical function is to prevent current flow. These materials have conductivities of the order of nanosiemens/meter. Most insulating materials have nonlinear properties, being good insulators at sufficiently low electric field intensities and temperatures but breaking down at higher field strengths and temperatures. Figure 2-7 shows the energy levels of different materials. See Sec. 4 for extensive tabulations of insulating properties.

Gaseous Conduction. A gas is usually a good insulator until it is ionized, which means that electrons are removed from molecules. The electrons are then available for conduction. Ionized gases are good conductors. Ionization can occur through raising temperature, bringing the gas into contact with glowing metals, arcs, or flames, or by an electric current.

Electrolytes. In liquid chemical compounds known as *electrolytes*, the passage of an electric current is accompanied by a chemical change. Atoms of metals and hydrogen travel through the liquid in the direction of positive current, while oxygen and acid radicals travel in the direction of electron current. Electrolytic conduction is discussed fully in Sec. 24.

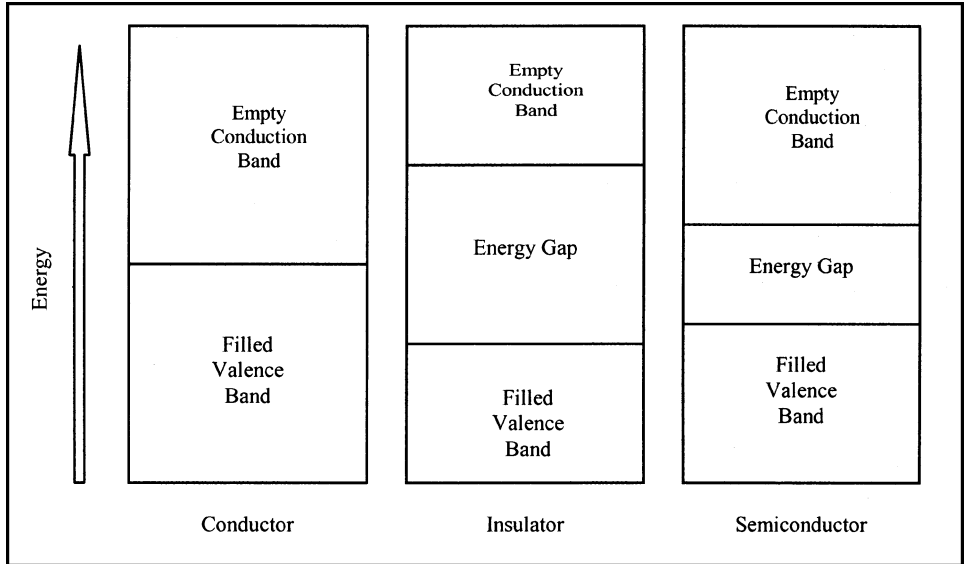


FIGURE 2-7 Component energy levels.

2.1.5 Resistors and Resistance

Resistors. A resistor is an electrical component or device designed explicitly to have a certain magnitude of resistance, expressed in ohms. Further, it must operate reliably in its environment, including electric field intensity, temperature, humidity, radiation, and other effects. Some resistors are designed explicitly to convert electric energy to heat energy. Others are used in control circuits, where they modify electric signals and energy to achieve desired effects. Examples include motor-starting resistors and the resistors used in electronic amplifiers to control the overall gain and other characteristics of the amplifier. A picture of a resistor may be seen in Fig. 2-8.

Ohm's Law. When the current in a conductor is steady and there are no voltages within the conductor, the value of the voltage v between the terminals of the conductor is proportional to the current i , or

$$v = ri \quad (2-19)$$

An example of Ohm's law may be seen in Fig. 2-9, where the coefficient of proportionality r is called the *resistance* of the conductor. The same law may be written in the form

$$i = gv \quad (2-20)$$

where the coefficient of proportionality $g = 1/r$ is called the *conductance* of the conductor. When the current is measured in amperes and the voltage in volts, the resistance r is in ohms and g is in siemens (often called mhos for reciprocal ohms). The phase of a resistor may be seen in Fig. 2-10.



FIGURE 2-8 Resistor.

2-10 SECTION TWO

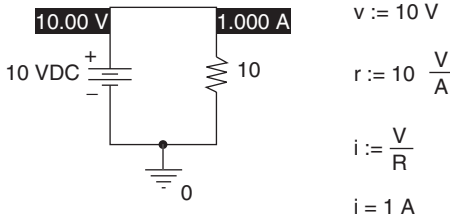


FIGURE 2-9 Ohm's law.

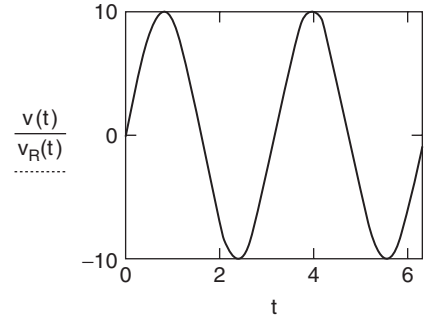


FIGURE 2-10 Phase of resistor.

Cylindrical Conductors. For current directed along the axis of the cylinder, the resistance r is proportional to the length l and inversely proportional to the cross section A , or

$$r = \rho \frac{l}{A} \quad (2-21)$$

where the coefficient of proportionality ρ (rho) is called the *resistivity* (or *specific resistance*) of the material. For numerical values of ρ for various materials, see Sec. 4.

The conductance of a cylindrical conductor is

$$g = \sigma \frac{A}{l} \quad (2-22)$$

where σ (sigma) is called the *conductivity* of the material. Since $g = 1/r$, the relation also holds that

$$\sigma = \frac{1}{\rho} \quad (2-23)$$

Changes of Resistance with Temperature. The resistance of a conductor varies with the temperature. The resistance of metals and most alloys increases with the temperature, while the resistance of carbon and electrolytes decreases with the temperature.

For usual conditions, as for about 100°C change in temperature, the resistance at a temperature t_2 is given by

$$R_{t_2} = R_{t_1} [1 + \alpha_{t_1} (t_2 - t_1)] \quad (2-24)$$

where R_{t_1} is the resistance at an initial temperature t_1 , and α_{t_1} is called the *temperature coefficient of resistance* of the material for the initial temperature t_1 . For copper having a conductivity of 100% of the International Annealed Copper Standard, $\alpha_{20} = 0.00393$, where temperatures are in degree Celsius (see Sec. 4).

An equation giving the same results as Eq. (2-24), for copper of 100% conductivity, is

$$\frac{R_{t_2}}{R_{t_1}} = \frac{234.4 + t_2}{234.4 + t_1} \quad (2-25)$$

where -234.4 is called the *inferred absolute zero* because if the relation held (which it does not over such a large range), the resistance at that temperature would be zero. For hard-drawn copper of 97.3% conductivity, the numerical constant in Eq. (2-25) is changed to 241.5. See Sec. 4 for values of these numerical constants for copper, and for other metals, see Sec. 4 under the metal being considered.



FIGURE 2-11 Inductor.

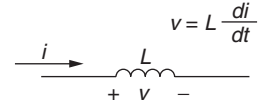


FIGURE 2-12 Inductor model and defining equation.

For 100% conductivity copper,

$$a_{t_1} = \frac{1}{234.4 + t_1} \quad (2-26)$$

When R_{t_1} and R_{t_2} have been measured, as at the beginning and end of a heat run, the “temperature rise by resistance” for 100% conductivity copper is given by

$$t_2 - t_1 = \frac{R_{t_2} - R_{t_1}}{R_{t_1}} (234.4 + t_1) \quad (2-27)$$

2.1.6 Inductors and Inductance

Inductors. An inductor is a circuit element whose behavior is described by the fact that it stores electromagnetic energy in its magnetic field. This feature gives it many interesting and valuable characteristics. In its most elementary form, an inductor is formed by winding a coil of wire, often copper, around a form that may or may not contain ferromagnetic materials. In this section, the behavior of the device at its terminal is discussed. Later, in the sections, on magnetic circuits, the device itself will be discussed. A picture of an inductor may be seen in Fig. 2-11.

Inductance. The property of the inductor that is useful in circuit analysis is called *inductance*. Inductance may be defined by either of the following equations:

$$v = L \frac{di}{dt} \quad i = \frac{1}{L} \int_0^t v(\tau) d\tau + i(0) \quad (2-28)$$

or

$$W = \frac{1}{2} Li^2 \quad (2-29)$$

where L = coefficient of self-inductance

i = current through the coil of wire

v = voltage across the inductor terminals

W = energy stored in the magnetic field

Figure 2-12 shows the symbol for an inductor and the voltage-current relationship for the device. The unit of inductance is called the *henry* (H), in honor of American physicist Joseph Henry.

The phase of an inductor may be seen in Fig. 2-13.

Mutual Inductance. If two coils are wound on the same coil form, or if they exist in close proximity, then a changing current in one coil will induce a voltage in the second coil. This effect forms the basis for transformers, one of the most pervasive of all electrical

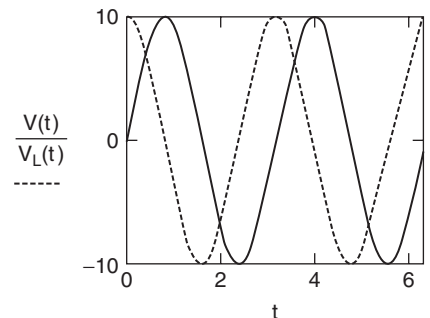


FIGURE 2-13 Phase of inductor.

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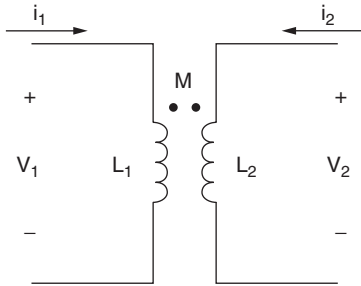


FIGURE 2-14 Mutual inductance model.

devices in use. Figure 2-14 shows the symbolic representation of a pair of coupled coils. The dots represent the direction of winding of the coils on the coil form in relation to the current and voltage reference directions. The equations become

$$\begin{aligned} v_1 &= L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \\ v_2 &= M \frac{di_1}{dt} + L_2 \frac{di_2}{dt} \end{aligned} \quad (2-30)$$

Mutual inductance also can be a source of problems in electrical systems. One example is the problem, now largely solved, of cross talk from one telephone line to another.

2.1.7 Capacitors and Capacitance

Charge Storage. A capacitor is a circuit element that is described through its principal function, which is to store electric energy. This property is called *capacitance*. In its simplest form, a capacitor is built with two conducting plates separated by a dielectric. A picture of a conductor may be seen in Fig. 2-15. Figure 2-16 shows the two usual symbols for a capacitor and the defining directions for voltage and current. These equations further describe the capacitor.

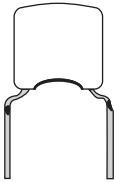


FIGURE 2-15 Capacitor.

or

$$i = C \frac{dv}{dt} \quad v(t) = \frac{1}{C} \int_0^t i(\tau) d\tau + v(0) \quad (2-31)$$

$$W = \frac{1}{2} C v^2 \quad (2-32)$$

where W = energy stored in the capacitor
 τ = dummy variable representing time
 C = capacitance in farads

The unit of capacitance is the farad (F), named in honor of English physicist Michael Faraday. The phase of a capacitor may be seen in Fig. 2-17.

2.1.8 Power and Energy

Power. The power delivered by an electrical source to an electrical device is given by

$$p(t) = v(t)i(t) \quad (2-33)$$

where p = power delivered
 v = voltage across the device
 i = current delivered to the device

The choice of algebraic sign is important. See Fig. 2-18a.

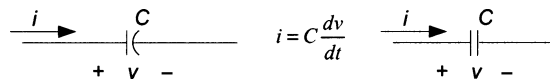


FIGURE 2-16 Capacitor—two symbols and defining equation.

If the device is a resistor, then the power delivered to the device is

$$p(t) = v(t)i(t) = i^2(t)R = \frac{v^2(t)}{R} \quad (2-34)$$

an equation known as *Joule's law*. In SI units, the unit of power is the watt (W), in honor of eighteenth century Scottish engineer James Watt.

Energy. The energy delivered by an electrical source to an electrical device is given by

$$W = \int_{t_1}^{t_2} v(t)i(t)dt \quad (2-35)$$

where the times t_1 and t_2 represent the starting and ending times of the energy delivery. In SI units, the unit of energy is the joule (J), in honor of English physicist James Joule. Power and energy are also related by the equation

$$p(t) = \frac{dW(t)}{dt} \quad (2-36)$$

A commonly used unit for electric energy measurement is a kilowatthour (kWh), which is equal to 3.6×10^6 joules.

Energy Density and Power Density. At times it is useful to evaluate materials and media by comparing their energy storage capability on a unit volume basis. The SI unit is joules per cubic meter, though conversion to other convenient combinations of units is possible. Power density is often an important consideration in, for example, heat or energy flow. The SI unit is watts per square meter, although any convenient unit system can be used.

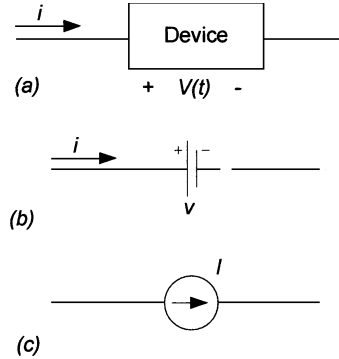


FIGURE 2-18 (a) Electrical device with definitions of voltage and current directions; (b) constant (D.C.) voltage source; (c) constant (D.C.) current source.

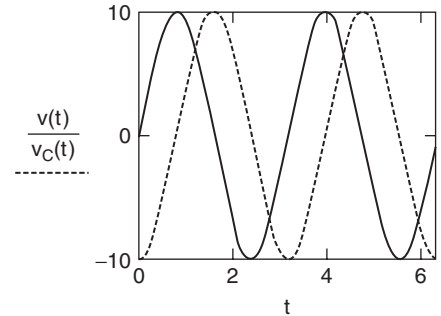


FIGURE 2-17 Phase of capacitor.

2.1.9 Physical Laws for Electric and Magnetic Circuits

Maxwell's Equations. Throughout much of the nineteenth century, engineers and physicists developed the theories that describe electricity and magnetism and their interrelations. In contemporary vector calculus notation, four equations can be written to describe the basic theory of electromagnetic fields. Collectively, they are known as *Maxwell's equations*, recognizing the work of James Clerk Maxwell's, a Scottish physicist, who solidified the theory. (Some writers consider only the first two as Maxwell's equations, calling the last two as supplementary equations.) The following symbols will be used in the description of Maxwell's equations:

E electric field intensity	vol (or V) enclosed volume in space
D electric flux density	L length of boundary around a surface
H magnetic field intensity	ρ electric charge density per unit volume
B magnetic flux density	J electric current density

Faraday's Law. Faraday observed that a time-varying magnetic field develops a voltage that can be observed and measured. This law is the basis for inductors. One common form of expressing the

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law is the equation

$$v = - \frac{\partial \phi}{\partial t}$$

where v is the voltage induced by the changing flux. The negative sign expresses the principle of conservation of energy, indicating that the direction of the voltage is such as to oppose the changing flux. This effect is known as *Lenz's law*.

In vector calculus notation, Faraday's law can be written in both integral and differential form. In integral form, the equation is

$$\oint E dL = \int_s \frac{\partial B}{\partial t} dS$$

where the line integral completely encircles the surface over which the surface integral is taken. In differential (point) form, Faraday's law becomes

$$\nabla \times E = - \frac{\partial B}{\partial t} \quad (2-37)$$

Ampere's Law. French physicist André-Marie Ampère developed the relation between magnetic field intensity and electric current that is a dual of Faraday's law. The current consists of two components, a steady or constant component and a time-varying component usually called *displacement current*. In vector calculus notation, Ampere's law is written first in integral form and then in differential form:

$$\oint H dL = I + \int_s \frac{dD}{dt} dS \quad (2-38)$$

$$\nabla \times H = J + \frac{\partial D}{\partial t} \quad (2-39)$$

An illustration of Ampere's law may be seen in Fig. 2-19. For more information, please refer to the Web site <http://www.ee.byu.edu/em/amplaw2.htm>.

Gauss's Law. Carl F. Gauss, a German physicist, stated the principle that the displacement current flowing over the surface of a region (volume) in space is equal to the charge enclosed. In integral and differential form, respectively, this law is written

$$\oint_s D dS = \int_{\text{vol}} \rho dv \quad (2-40)$$

$$\nabla \cdot D = \rho \quad (2-41)$$

For further study, please refer to the Web site <http://www.ee.byu.edu/ee/em/eleclaw.htm>.

An illustration of Gauss's law may be seen in Fig. 2-20.

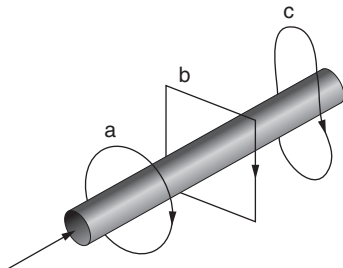


FIGURE 2-19 Ampere's law illustration.

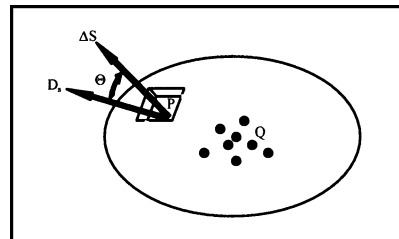


FIGURE 2-20 Gauss's law illustration.

Gauss's Law for Magnetics. One of the postulates of electromagnetism is that there are no free magnetic charges but these charges always exist in pairs. While searches are continually being made, and some claims of discovery of free charges have been made, the postulate is still adequate to explain observations in cases of interest here. A consequence of this postulate is that, for magnetics, Gauss's law become

$$\oint_s B dS = 0 \quad (2-42)$$

$$\nabla \cdot B = 0 \quad (2-43)$$

Kirchhoff's Laws. In the analysis and design of electric circuits, a fundamental principle implies that the dimensions are small. This means that it is possible to neglect the spatial variations in electromagnetic quantities. Another way of saying this is that the dimensions of the circuit are small compared with the wavelengths of the electromagnetic quantities and thus that it is necessary to consider only time variations. This means that Maxwell's equations, which are partial integrodifferential equations, become ordinary integrodifferential equations in which the independent variable is time, represented by t .

Kirchhoff's Current Law. The assumption of small dimensions means that no free electric charges can exist in the region in which a circuit is being analyzed. Thus, Gauss's law (in integral form) becomes

$$\sum i = 0 \quad (2-44)$$

at any point in the circuit. The points of interest usually will be *nodes*, points at which three or more wires connect circuit elements together. This law will be abbreviated KCL and was enunciated by German physicist, Gustav Robert Kirchhoff. It is one of the two fundamental principles of circuit analysis. Figure 2-21 shows a sample circuit simulated in PSPICE. We can see the current flowing through the $3\ \Omega$ resistor (3.5 A) is equal to the sum of the current flowing through the $6\ \Omega$ resistor (1.5 A) and the current flowing through the $1.5\ \Omega$ resistor (2 A).

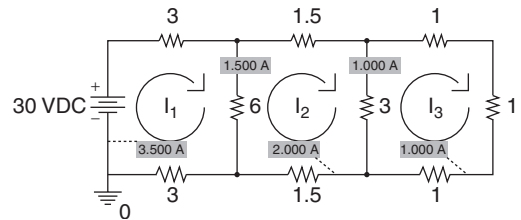


FIGURE 2-21 Kirchhoff's current law.

Kirchhoff's Voltage Law. The second fundamental principle, abbreviated KVL, follows from applying the assumption of small size to Faraday's law in integral form. Since the circuit is small, it is possible to take the surface integral of magnetic flux density as zero and then to state that the sum of voltages around any closed path is zero. In equation form, it can be written as

$$\sum v = 0 \quad (2-45)$$

Figure 2-22 shows a sample circuit simulated in PSPICE. We have

$$\sum V = V_{0 \rightarrow v_1} + V_{v_1 \rightarrow v_2} + V_{v_2 \rightarrow v_7} + V_{v_7 \rightarrow 0} = -30 + 10.5 + 9 + 10.5 = 0$$

2.1.10 Electric Energy Sources and Representations

Sources. In circuit analysis, the goal is to start with a connected set of circuit elements such as resistors, capacitors, operational amplifiers, and other devices, and to find the voltages across and currents through each element, additional quantities, such as power dissipated, are often computed. To energize the circuit, sources of electric energy must be connected. Sources are modeled in various ways.

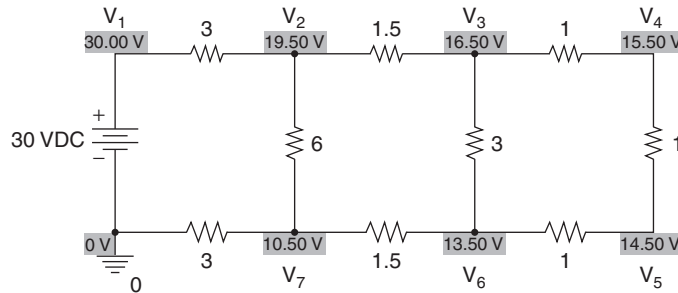


FIGURE 2-22 Kirchhoff's voltage law.

One convenient classification is to consider constant (dc) sources, sinusoidal (ac) sources, and general time-varying sources. The first two are of interest in this section.

DC Sources. Some sources, such as batteries, deliver electric energy at a nearly constant voltage, and thus they are modeled as constant voltage sources. The term *dc sources* basically means *direct-current sources*, but it has come to stand for constant sources as well. Figure 2-23 shows the standard symbol for a dc source. Other sources are modeled as dc current (or constant-current) sources. Figure 2-18*b* and *c* show the symbols used for these models.

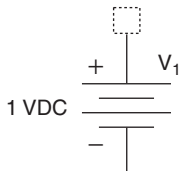


FIGURE 2-23 D.C. source.

AC Sources. Most of the electric energy used in the world is generated, distributed, and utilized in sinusoidal form. Thus, beginning with Charles P. Steinmetz, a German-American electrical engineer, much effort has been devoted to finding efficient ways to analyze and design circuits that operate under sinusoidal excitation conditions. Sources of this type are frequently called ac (for alternating current) sources. Figure 2-24 shows the standard symbol for an ac source. The most general expression for a voltage in sinusoidal form is of the type

$$v(t) = V_m \cos(2\pi ft + \alpha) = V_m \cos(\omega t + \alpha) \quad (2-46)$$

and, for a current

$$i(t) = I_m \cos(2\pi ft + \beta) = I_m \cos(\omega t + \beta) \quad (2-47)$$

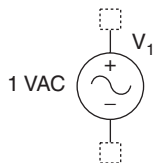


FIGURE 2-24 AC source.

Some writers use sine functions instead of cosine functions, but this has only the effect of changing the angles α and β .

These expressions have three identifying characteristics, the maximum or peak value (V_m or I_m), the phase angle (α or β), and the frequency [f , measured in hertz (Hz) or cycles per second, or ω , measured in radians/second]. A powerful method of circuit analysis depends on these observations. It is called *phasor analysis*.

2.1.11 Phasor Analysis

The Imaginary Operator. A term that arises frequently in phasor analysis is the imaginary operator

$$j = \sqrt{-1} \quad (2-48)$$

(Electrical engineers use j , since i is reserved as the symbol for current. Mathematicians, physicists, and others are more likely to use i for the imaginary operator.)

Euler's Relation. A relationship between trigonometric and exponential functions, known as *Euler's relation*, plays an important role in phasor analysis. The equation is

$$e^{jx} = \cos x + j \sin x \quad (2-49)$$

If this equation is solved for the trigonometric terms, the result is

$$\cos x = \frac{e^{jx} + e^{-jx}}{2} \quad (2-50)$$

$$\sin x = \frac{e^{jx} - e^{-jx}}{2} \quad (2-51)$$

In phasor analysis, this equation is used by writing it as

$$e^{j(\omega t + \alpha)} = e^{j\omega t} e^{j\alpha} = \cos(\omega t + \alpha) + j \sin(\omega t + \alpha) \quad (2-52)$$

Thus, it is observed that the cosine term in the preceding expressions for voltage and current is equal to the real-part term from Euler's relation. Thus, it will be seen possible to substitute the general exponential term for the cosine term in the source expressions, then, to find the solution (currents and voltages) to the exponential excitation, and finally, to take the real part of the result to get the final answer.

Steady-State Solutions. When the complete solution for current and voltage in a linear, stable, time-invariant circuit is found, two types of terms are found. One type of term, called the *complementary function* or *transient solution*, depends only on the elements in the circuit and the initial energy stored in the circuit when the forcing function is connected. If the circuit is stable, this term typically becomes very small in a short time.

The second type of term, called the *particular integral* or *steady-state solution*, depends on the circuit elements and configuration and also on the forcing function. If the forcing function is a single-frequency sinusoidal function, then it can be shown that the steady-state solution will contain terms at this same frequency but with differing amplitudes and phases. The goal of phasor analysis is to find the amplitudes and phases of the voltages and currents in the solution as efficiently as possible, since the frequency is known to be the same as the frequency of the forcing function.

Definition of a Phasor. The phasor representation of a sinusoidal function is defined as a complex number containing the amplitude and phase angle of the original function. Specifically, if

$$v(t) = V_m \cos(\omega t + \alpha) = V_m \sin(\omega t + \alpha + \pi/2) \quad (2-53)$$

then the phasor representation is given by

$$V = V_m e^{j\alpha} \quad (2-54)$$

A phasor can be converted to a sinusoidal time function by using the definition in reverse. See Sec. 2.1.12 for an alternative definition of a phasor, which differs only by a multiplicative constant.

Phasor Algebra. It is necessary at times to perform arithmetic and algebraic operations on phasors. The rules of phasor algebra are identical with those of complex number algebra and vector algebra. Specifically,

$$\begin{aligned} V_1 e^{j\alpha_1} \pm V_2 e^{j\alpha_2} &= (V_1 \cos \alpha_1 \pm V_2 \cos \alpha_2) + j(V_1 \sin \alpha_1 \pm V_2 \sin \alpha_2) \\ (V_1 e^{j\alpha_1})(V_2 e^{j\alpha_2}) &= (V_1 V_2) e^{j(\alpha_1 + \alpha_2)} \\ \frac{V_1 e^{j\alpha_1}}{V_2 e^{j\alpha_2}} &= \frac{V_1}{V_2} e^{j(\alpha_1 - \alpha_2)} \end{aligned} \quad (2-55)$$

A few examples will show the calculations. In the examples, the angles are expressed in radians. Sometimes degrees are used instead, at the option of the analyst.

$$\begin{aligned}
 5e^{j\pi/5} + 4e^{j2\pi/3} &= (4.05 + j2.94) + (-2.00 + j3.46) = 6.72e^{j1.26} \\
 (8e^{j\pi/4})(1.3e^{j3\pi/5}) &= 10.4e^{j2.67} = -9.27 + j4.72
 \end{aligned}
 \tag{2-56}$$

With a modern electronic calculator or one of many suitable computer programs, it is possible to perform these calculations readily, though they may appear tedious.

Integration and Differentiation Operations. Let a phasor be represented by

$$P_1 = P_m e^{j\omega t} e^{j\theta} \tag{2-57}$$

where the frequency is included for completeness. Differentiation and integration become, respectively,

$$\frac{dP_1}{dt} = j\omega P_m e^{j\omega t} e^{j\theta} = \omega P_m e^{j\omega t} e^{j(\theta+90^\circ)} \tag{2-58}$$

$$\int P_1 dt = \frac{1}{j\omega} P_m e^{j\omega t} e^{j\theta} = \frac{1}{\omega} P_m e^{j\omega t} e^{j(\theta-90^\circ)} \tag{2-59}$$

Reactance and Susceptance. For an inductor, the ratio of the phasor voltage to the phasor current is given by $j\omega L$. This quantity is called the *reactance* of the inductor, and its reciprocal is called *susceptance*. For a capacitor, the ratio of phasor voltage to phasor current is $1/(j\omega C) = -j(\omega C)$. This quantity is called the *reactance* of a capacitor. Its reciprocal is called *susceptance*. The usual symbol for reactance is X , and for susceptance, B .

Impedance and Admittance. Analysis of ac circuits requires the analyst to replace each inductor and capacitor with appropriate susceptances or reactances. Resistors and constant controlled sources are unchanged. Application of any of the methods of circuit analysis will lead to a ratio of a voltage phasor to a current phasor. This ratio is called *impedance*. It has a real (or resistive) part and an imaginary (or reactive) part. Its reciprocal is called *admittance*. The real part of admittance is called the *conductive* part, and the imaginary part is called the *susceptive part*. In Sec. 2.1.15, an analysis of a circuit shows the use of these ideas.

2.1.12 AC Power and Energy Considerations

Effective or RMS Values. If a sinusoidal current $i(t) = I_m \cos(\omega t + \alpha)$ flows through a resistor of $R \Omega$, then, over an integral number of cycles, the average power delivered to the resistor is found to be

$$P_{\text{avg}} = \frac{I_m^2}{2} R \tag{2-60}$$

This amount of power is identical to the amount of power that would be delivered by a constant (dc) current of $I_m/\sqrt{2}$ amperes. Thus, the effective value of an ac current (or voltage) is equal to the maximum value divided by $\sqrt{2}$.

The effective value is commonly used to describe the requirements of ac systems. For example, in North America, rating a light bulb at 120 V implies that the bulb should be used in a system where the effective voltage is 120 V. In turn, the voltages and currents quoted for distribution systems are effective values.

An alternative term is root-mean-square (rms) value. This term follows from the formal definition of effective or rms values of a function,

$$F_{\text{rms}} = \sqrt{\frac{1}{T} \int_{t_0}^{t_0+T} (f(t))^2 dt} \quad (2-61)$$

Frequently, when phasor ideas are being used, effective rather than peak values are implied. This is quite common in electric power system calculations, and it is necessary for the engineer simply to determine which is being used and to be consistent.

Power Factor. When the voltage across a device and the current through a device are given, respectively, by

$$v(t) = V_m \cos(\omega t + \alpha) \quad (2-62)$$

and

$$i(t) = I_m \cos(\omega t + \beta) \quad (2-63)$$

a computation of the average power over an integral number of cycles gives

$$P_{\text{avg}} = \frac{V_m I_m}{2} \cos(\alpha - \beta) \quad (2-64)$$

and

$$P_{\text{avg}} = V_{\text{eff}} I_{\text{eff}} \cos(\alpha - \beta) \quad (2-65)$$

The angle $(\alpha - \beta)$, which is the phase difference between the voltage and current, is called the *power factor angle*. The cosine of the angle is called the *power factor* because it represents the ratio of the average power delivered to the product of voltage and current.

Reactive Voltamperes. When the voltage across a device and the current through a device are given, respectively, by

$$v(t) = V_m \cos(\omega t + \alpha) \quad (2-66)$$

and

$$i(t) = I_m \cos(\omega t + \beta) \quad (2-67)$$

a computation of the power delivered to the device as a function of time shows

$$p(t) = \frac{V_m I_m}{2} [\cos(\alpha - \beta) + \cos(2\omega t + \alpha + \beta)] \quad (2-68)$$

In addition to the constant term that represents the average power, there is a double-frequency term that represents energy that is interchanged between the electric and magnetic fields of the device and the source. This quantity is called by the term *reactive voltamperes* (vars). It may be shown that

$$\text{var} = \frac{V_m I_m}{2} \sin(\alpha - \beta) \quad (2-69)$$

and

$$\text{var} = V_{\text{eff}} I_{\text{eff}} \sin(\alpha - \beta) \quad (2-70)$$

Power and Vars. If the phasor voltage across a device and the phasor current through the device are given, respectively, by

$$V_1 = V_{\text{eff}} e^{j\alpha} \quad (2-71)$$

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and

$$I_2 = I_{\text{eff}} e^{j\beta} \quad (2-72)$$

then the expression

$$VA = V_{\text{eff}} I_{\text{eff}}^* = V_{\text{eff}} I_{\text{eff}} [\cos(\alpha - \beta) + j \sin(\alpha - \beta)] \quad (2-73)$$

where * represents the complex conjugate, which may be used to find both average power and vars. The real part of the expression is the average power, while the imaginary part is the vars.

2.1.13 Controlled Sources

Models. When circuits containing electronic devices such as amplifiers and similar devices are analyzed or designed, it is necessary to have a linear circuit model for the electronic device. These devices have a minimum of three terminals. Currents flow between terminal pairs, and voltages appear across terminal pairs. One terminal may not be common to both pairs. A useful model is provided by a controlled source. Four such models may be distinguished, as shown in Fig. 2-25. Examples of use will appear in the paragraphs on circuit analysis.

Voltage-Controlled Voltage Source (VCVS). If the voltage across one terminal pair is proportional to the voltage across a second terminal pair, then the model of Fig. 2-26a may be used. In this model, the output voltage v_{xz} is proportional to the input voltage $v_{yz'}$ with a proportionality constant A . It should be noted, however, that this device is not usually reciprocal, that is, impression of a voltage at the terminals xz will not lead to a voltage at terminals yz' .

Voltage-Controlled Current Source. If the current flow between a terminal pair is proportional to the voltage across another pair, then the appropriate model is a voltage-controlled current source (VCCS). See Fig. 2-26b.

Current-Controlled Current Source. If the current flow between a terminal pair is proportional to the current through another terminal pair, then the appropriate model is a current-controlled current source (CCCS), as shown in Fig. 2-26c.

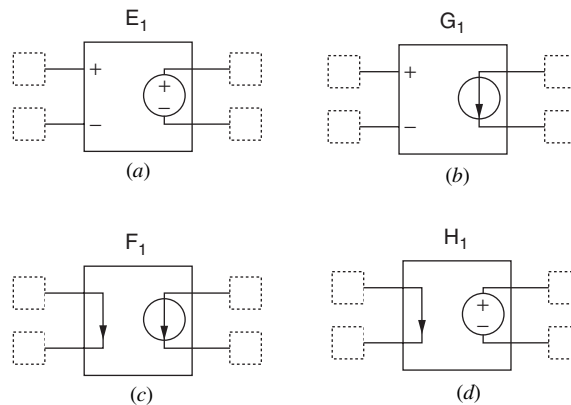


FIGURE 2-25 (a) Voltage-controlled voltage source; (b) Voltage-controlled current source; (c) Current-controlled current source; (d) Current-controlled voltage source.

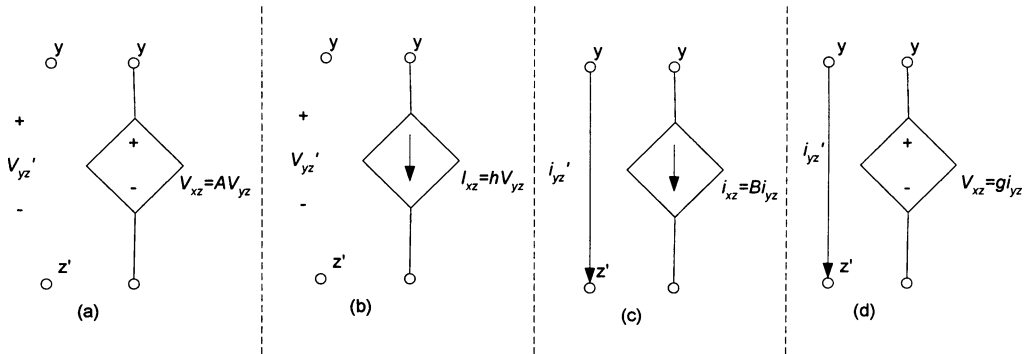


FIGURE 2-26 Controlled sources: (a) voltage-controlled voltage source; (b) voltage-controlled current source; (c) current-controlled current source; (d) current-controlled voltage source.

Current-Controlled Voltage Source. If the voltage across one terminal pair is proportional to the current flow through another terminal pair, then the appropriate model is a current-controlled voltage source (CCVS), as shown in Fig. 2-26d.

2.1.14 Methods for Circuit Analysis

Circuit Reduction Techniques. When a circuit analyst wishes to find the current through or the voltage across one of the elements that make up a circuit, as opposed to a complete analysis, it is often desirable to systematically replace elements in a way that leaves the target elements unchanged, but simplifies the remainder in a variety of ways. The most common techniques include series/parallel combinations, wye/delta (or tee/pi) combinations, and the Thevenin-Norton theorem.

Series Elements. Two or more electrical elements that carry the same current are defined as being in series. Figure 2-27 shows a variety of equivalents for elements connected in series.

Parallel Elements. Two or more electrical elements that are connected across the same voltage are defined as being in parallel. Figure 2-28 shows a variety of equivalents for circuit elements connected in parallel.

Wye-Delta Connections. A set of three elements may be connected either as a wye, shown in Fig. 2-29a, or a delta, shown in Fig. 2-29b. These are also called *tee* and *pi* connections, respectively. The equations give equivalents, in terms of resistors, for converting between these connection forms.

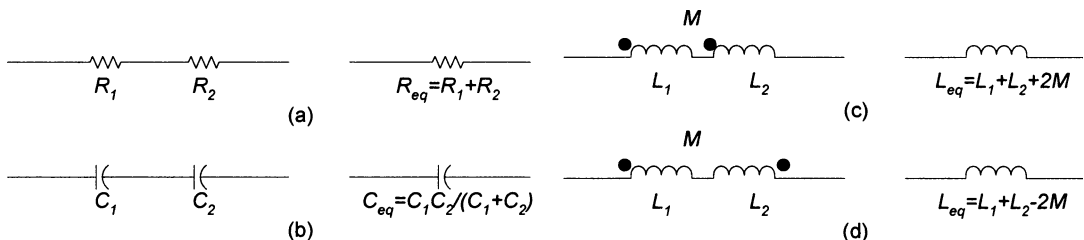


FIGURE 2-27 Series-connected elements and equivalents: (a) resistors in series; (b) capacitors in series; (c) inductors in series, aiding fluxes; (d) inductors in series, opposing fluxes.

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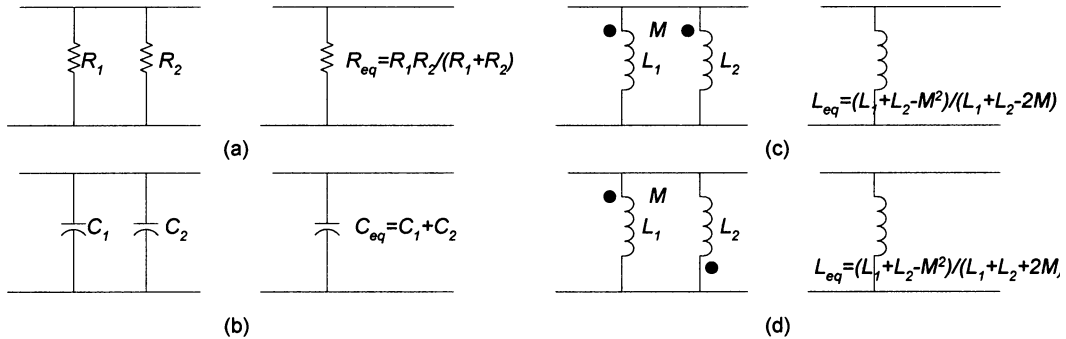


FIGURE 2-28 Parallel-connected elements and equivalents: (a) resistors in parallel; (b) capacitors in parallel; (c) inductors in parallel, aiding fluxes; (d) inductors in parallel, opposing fluxes.

$$R_c = \frac{R_1 R_2 + R_1 R_3 + R_2 R_3}{R_1}$$

$$R_b = \frac{R_1 R_2 + R_1 R_3 + R_2 R_3}{R_2} \quad (2-74)$$

$$R_a = \frac{R_1 R_2 + R_1 R_3 + R_2 R_3}{R_3}$$

$$R_1 = \frac{R_a R_b}{R_a + R_b + R_c}$$

$$R_2 = \frac{R_a R_c}{R_a + R_b + R_c} \quad (2-75)$$

$$R_3 = \frac{R_b R_c}{R_a + R_b + R_c}$$

In practice, application of one of these conversion pairs will lead to additional series or parallel combinations that can be further simplified.

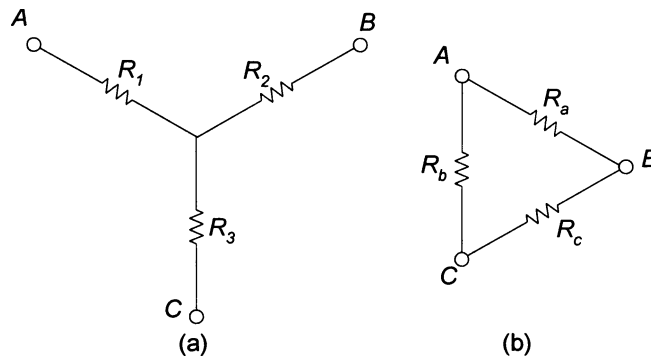


FIGURE 2-29 (a) Wye-connected elements; (b) delta-connected elements.

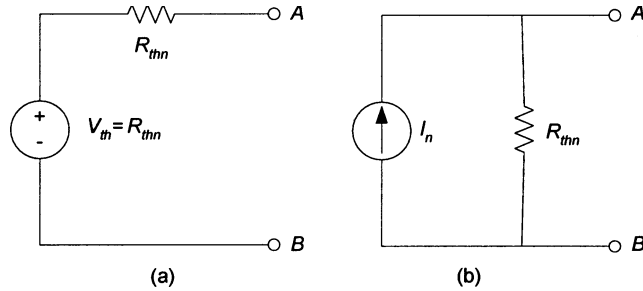


FIGURE 2-30 (a) Thevenin equivalent circuit model; (b) Norton equivalent circuit model.

Thevenin-Norton Theorem. The Thevenin theorem and its dual, the Norton theorem, provide the engineer with a convenient way of characterizing a network at a terminal pair. The method is most useful when one is considering various loads connected to a pair of output terminals. The equivalent can be determined analytically, and in some cases, experimentally. Terms used in these paragraphs are defined in Fig. 2-30.

Thevenin Theorem. This theorem states that at a terminal pair, any linear network can be replaced by a voltage source in series with a resistance (or impedance). It is possible to show that the voltage is equal to the voltage at the terminal pair when the external load is removed (open circuited), and that the resistance is equal to the resistance calculated or measured at the terminal pair with all independent sources de-energized. De-energization of an independent source means that the source voltage or current is set to zero but that the source resistance (impedance) is unchanged. Controlled (or dependent) sources are not changed or de-energized.

Norton Theorem. This theorem states that at a terminal pair, any linear network can be replaced by a current source in parallel with a resistance (or impedance). It is possible to show that the current is equal to the current that flows through the short-circuited, terminal pair when the external load is short circuited, and that the resistance is equal to the resistance calculated or measured at the terminal pair with all independent sources de-energized. De-energization of an independent source means that the source voltage or current is set to zero but that the source resistance (impedance) is unchanged. Controlled (or dependent) sources are not changed or de-energized.

Thevenin-Norton Comparison. If the Thevenin equivalent of a circuit is known, then it is possible to find the Norton equivalent by using the equation

$$V_{th} = I_n R_{thn} \quad (2-76)$$

as indicated in Fig. 2-30.

Thevenin-Norton Example. Figure 2-31a shows a linear circuit with a current source and a voltage-controlled voltage source. Figure 2-31b shows a calculation of the Thevenin or open-circuit voltage. Figure 2-31c shows a calculation of the Norton or short-circuit current. Figure 2-31d shows the final Norton and Thevenin equivalent circuits.

2.1.15 General Circuit Analysis Methods

Node and Loop Analysis. Suppose b elements or branches are interconnected to form a circuit. A complete solution for the network is one that determines the voltage across and the current through each element. Thus, $2b$ equations are needed. Of these, b are given by the voltage-current relations, for example, Ohm's law, for each element. The others are obtained from systematic application of Kirchhoff's voltage and current laws.

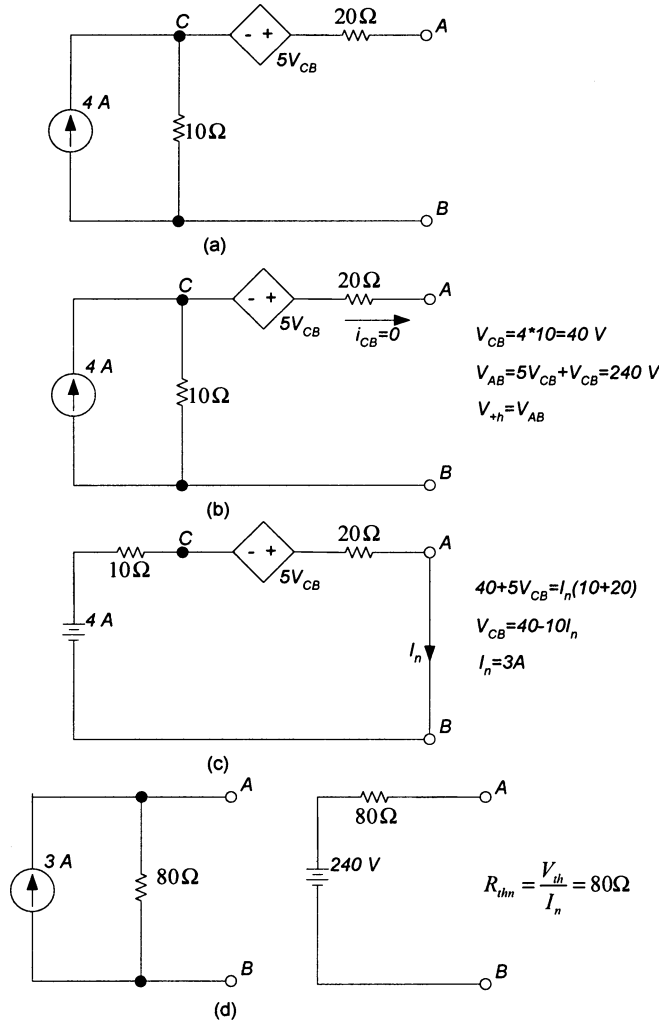


FIGURE 2-31 To illustrate Thevenin-Norton theorem: (a) example circuit; (b) calculation of Thevenin voltage; (c) calculation of Norton current; (d) Norton and Thevenin equivalent circuits.

Define a point at which three or more elements or branches are connected as a node (some writers call this an *essential node*). Suppose that the circuit has n such nodes or points. It is possible to write Kirchhoff's current law equations at each node, but one will be redundant, that is, it can be derived from the others. Thus, $n-1$ KCL equations can be written, and these are independent.

This means that to complete the analysis, it is necessary to write $[b-(n-1)]$ KVL equations, and this is possible, though care must be taken to ensure that they are independent. In practice, it is typical that either KCL or KVL equations are written, but not both. Sufficient information is usually available from either set. Which set is chosen depends on the analyst, the comparative number of equations, and similar factors.

Nodal Analysis. Figure 2-32a shows a typical node in a circuit that is isolated for attention. The voltage on this node is measured or calculated with a reference somewhere in the circuit, often but not always the node that is omitted in the analysis. Other nodes, including the reference node, are shown along with connecting elements. To illustrate the technique, five additional nodes are chosen, including the reference nodes. The boxes labeled Y are called *admittances*.

Kirchhoff's current law written at the node states that

$$i_{k-2} + i_{k-1} + i_k + i_{k+1} + i_{k+2} = I_{in,k} \quad (2-77)$$

An expression for each of the currents can be written

$$i_{k+1} = (V_k - V_{k+1})Y_{k+1} \quad (2-78)$$

When all the equations that can be written are written, collected, and organized into matrix format, the general result is

$$\begin{bmatrix} Y_{11} + Y_{12} + Y_{13} + \cdots & -Y_{12} & -Y_{13} & \cdots \\ -Y_{21} & Y_{21} + Y_{22} + Y_{23} + \cdots & -Y_{23} & \cdots \\ -Y_{31} & -Y_{32} & Y_{31} + Y_{32} + Y_{33} + \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ \vdots \end{bmatrix} \quad (2-79)$$

where the square matrix describes the circuit completely, the column matrix (vector) of voltages describes the dependent variables which are the node voltages, and the column matrix of currents describes the forcing function currents that enter each node.

Nodal Analysis with Controlled Sources. If a controlled source is present, it is most convenient to use the Thevenin-Norton theorem to convert the controlled source to a voltage-controlled current

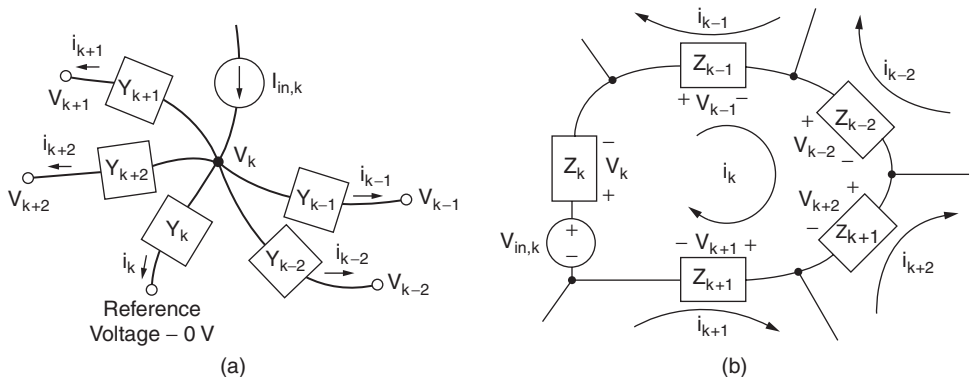


FIGURE 2-32 To illustrate node and loop analysis: (a) typical node isolated for study; (b) typical loop isolated for study.

source. When this is done, the right side of the preceding equation will contain the dependent variables (voltages) in addition to independent current sources. These voltage terms can then be transposed to the left side of the matrix equation. The result is the addition of terms in the circuit matrix that make the matrix nonsymmetric.

Solution of Nodal Equations. In dc and ac (sinusoidal steady-state) circuits, the Y terms are numerical terms. Calculators that handle matrices and mathematical software programs for computers permit rapid solutions. Ordinary determinant methods also suffice. The result will be a set of values for the various voltages, all determined with respect to the reference node voltage. If the terms in the equation are generalized admittances (see Sec. 2.1.20 on Laplace transform analysis), then the solution will be a quotient of polynomials in the Laplace transform variable s . More is said about such solutions in that section.

Loop Current Analysis. Define a *loop* as a closed path in a circuit and a *loop current* as a current that flows around this path. See Fig. 2-32b, which shows one loop that has been isolated for attention, the associated loop current, and loop currents that flow in neighboring loops. It is noted that the current through any given element is found to be the difference between two loop currents if the circuit is planar, that is, can be drawn on a flat surface without crossing wires. (If the circuit is nonplanar, the technique is still valid, but it can become more complex, and some element currents will be composed of three or more loop currents.) The elements labeled Z are called impedances.

Kirchhoff's voltage law written around the loop states that

$$v_{k-2} + v_{k-1} + v_k + v_{k+1} + v_{k+2} = V_{in,k} \quad (2-80)$$

An expression for each of the voltages can be written

$$v_{k+1} = (i_k - i_{k+1})Z_{k+1} \quad (2-81)$$

When all the equations that can be written are written, collected, and organized into matrix format, the general result is

$$\begin{bmatrix} Z_{11} + Z_{12} + Z_{13} + \cdots & -Z_{12} & -Z_{13} & \cdots \\ -Z_{21} & Z_{21} + Z_{22} + Z_{23} + \cdots & -Z_{23} & \cdots \\ -Z_{31} & -Z_{32} & Z_{31} + Z_{32} + Z_{33} + \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \end{bmatrix} \quad (2-82)$$

where the square matrix describes the circuit completely, the column matrix (vector) of currents describes the dependent variables which are the loop currents, and the column matrix of voltages describes the forcing function voltages that act in each loop.

Loop Current Analysis with Controlled Sources. If a controlled source is present, it is most convenient to use the Thevenin-Norton theorem to convert the controlled source to a current-controlled voltage source. When this is done, the right side of the preceding equation will contain the dependent variables (currents) in addition to independent voltage sources. These current terms can then be transposed to the left side of the matrix equation. The result is the addition of terms in the circuit matrix that make the matrix nonsymmetric.

Solution of Loop Current Equations. In D.C. and A.C. (sinusoidal steady-state) circuits, the Z terms are numerical terms. Calculators that handle matrices and mathematical software programs for computers facilitate the numerical work. Ordinary determinant methods also suffice. The result will be a set of values for the various loop currents, from which the actual element currents can be readily obtained. If the terms in the equation are generalized admittances (see Sec. 2.1.20 on Laplace transform analysis), then the solution will be a quotient of polynomials in the Laplace transform variable s . More is said about such solutions in those paragraphs.

Sinusoidal Steady-State Example. Figure 2-33 shows a circuit with a current source, two resistors, two capacitors, and one inductor. (The network is scaled.) The current source has a frequency of 2 rad/s and is sinusoidal. Figure 2-33b shows the circuit prepared for phasor analysis. The equations that follow show the writing of KCL equations for two voltages and their solution, which is shown as a phasor and as a time function.

$$2 = V_1 (1 + j2.00 - j0.25) - V_2 (-j0.25) \quad (2-83)$$

$$0 = -V_1 (-j0.25) + V_2 (1 + j2.00 - j0.25) \quad (2-84)$$

$$V_2 = 0.0615 - j0.1077 = 0.124e^{j(2.6224)} \text{ (angle in radians)} \quad (2-85)$$

$$V_2(t) = 0.1240 \cos(2t + 2.6224) = 0.1240 \cos(2t + 150.25^\circ) \quad (2-86)$$

Computer Methods. The rapid development of computers in the last few years has led to the development of many programs written for the purpose of analyzing electric circuits. Because of their rapid analysis capability, they also are effective in design of new circuits. Programs exist for personal

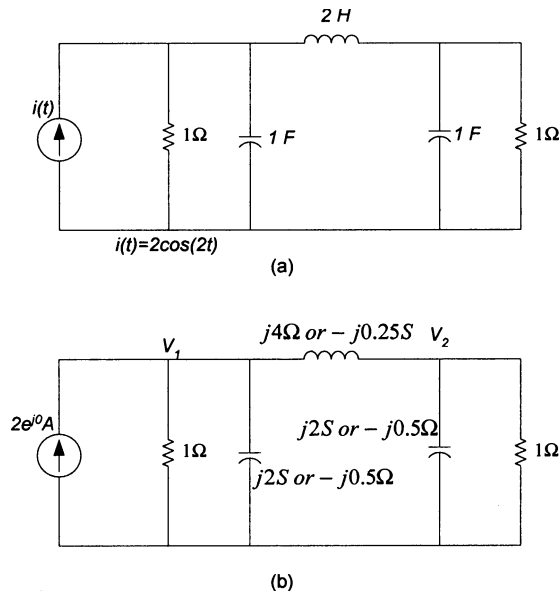


FIGURE 2-33 Sinusoidal steady-state analysis example: (a) circuit with sinusoidal source and two nodes; (b) phasor domain equivalent circuit.

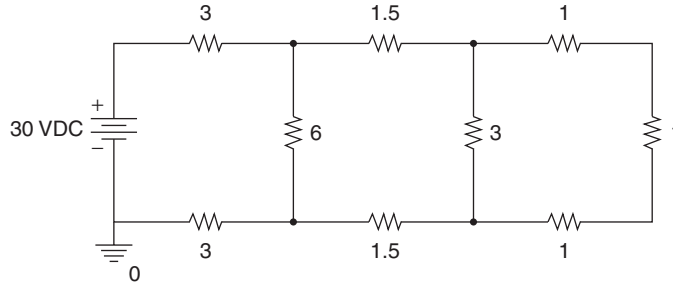


FIGURE 2-34 PSPICE circuit.

computers, minicomputers, and mainframe computers. Probably the most popular is SPICE, which is an acronym for Simulation Program with Integrated Circuit Emphasis. The personal computer version of this program is PSPICE. Most of these programs are in the public domain in the United States. It is convenient to discuss how a circuit is described to a computer program and what data are available in an analysis. Figures 2-34 and 2-35 show a sample PSPICE circuit.

SPICE Circuit Description. The analysis of a circuit with SPICE or another program requires the analyst to describe the circuit completely. Every node is identified, and each branch is described by type, numerical value, and nodes to which it is connected $e^{j(\alpha_1 + \alpha_2)}$. Active devices such as transistors and operational amplifiers can be included in the description, and the program library contains complete data for many commonly used electronic elements.

SPICE Analysis Results. The analyst has a lot of control over what analysis results are computed. If a circuit is resistive, then a D.C. analysis is readily performed. This analysis is easily expanded to do a sensitivity analysis, which is a consideration of how results change when certain components change. Further, such analyses can be done both for linear and nonlinear circuits.

If the analyst wishes, a sinusoidal steady-state analysis is then possible. This includes *small-signal analysis*, a consideration of how well circuits such as amplifiers amplify signals which appear as currents or voltages. A frequency response is possible, and the results may be graphed with a variety of independent variables.

Other possible analyses include noise analyses—a study of the effect of electrical noise on circuit performance—and distortion analyses. Still others include transient response studies, which are most important in circuit design. The results may be graphed in a variety of useful ways. References give useful information.

Numerical example for the small-signal analysis is shown in Fig. 2-36.

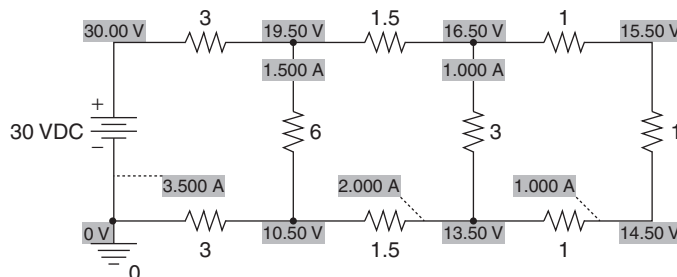


FIGURE 2-35 PSPICE analysis.

Additional Programs. A major advantage of these computer methods is that they work well for all types of circuit—low or high power, low or high frequency, power or communications. This is true even though the program's name might indicate otherwise. However, in some types of analysis, special programs have been developed to facilitate design and analysis. For example, power systems are often described by circuits that have more than 1000 nodes but very few nonzero entries in the circuit matrix. These special characteristics have led to the development of efficient programs for such studies. Several references address these issues.

2.1.16 Electric Energy Distribution in 3-Phase Systems

General Note. In most of the world, large amounts of electric energy are distributed in 3-phase systems. The reasons for this decision include the fact that such systems are more efficient than single-phase systems. In other words, they have reduced losses and use materials more efficiently. Further, it can be shown that a 3-phase system distributes electric power at a constant rate, not at the time-varying rate shown earlier for single-phase systems. It is convenient to consider balanced systems and unbalanced systems separately. Also, both loads and sources need to be considered.

Balanced 3-Phase Sources. A 3-phase source consists of three voltage sources that are sinusoidal, equal in magnitude, and differ in phase by 120° . Thus, the set of voltages shown below is a balanced 3-phase source, shown both in time and in phasor format.

$$\begin{aligned} v_{ab} &= V_m \cos(377t) & V_m e^{j0} \\ v_{bc} &= V_m \cos(377t - 120^\circ) & V_m e^{-j120^\circ} \\ v_{ca} &= V_m \cos(377t + 120^\circ) & V_m e^{+j120^\circ} \end{aligned} \quad (2-87)$$

(In these expressions, peak values have been used rather than effective values. Further, degrees and radians are mixed, which is commonly done for the sake of clarity and convention but which can lead to numerical errors in calculators and computers if not reconciled.) Note that the sum of the three voltages is zero.

These three sources may be connected in either of the two ways to form a balanced system. One is the wye (star or tee) connection and the other is the delta (mesh or pi) connection. Both are shown in Fig. 2-37. It is noted that in the wye connection, a fourth point is needed, which is labeled O . The terminals labeled

```

V_source := 30 V

R1 := 3 Ω   R4 := 1.5 Ω   R7 := 1 Ω
R2 := 6 Ω   R5 := 3 Ω    R8 := 1 Ω
R3 := 3 Ω   R6 := 1.5 Ω   R9 := 1 Ω

R_loop3 := R7 + R8 + R9

R_loop2 := R4 + R6 + 1 / (1/R5 + 1/R_loop3)

R_loop1 := R1 + R3 + 1 / (1/R2 + 1/R_loop2)

I := V_source / R_loop1

I = 3.5 A

```

FIGURE 2-36 MathCAD equation.

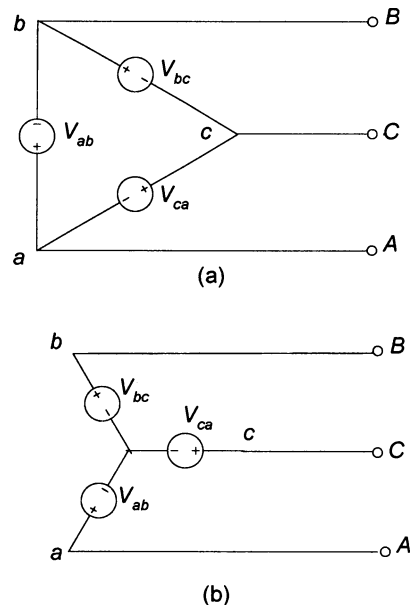


FIGURE 2-37 3-Phase source connections: (a) delta connection; (b) wye connection.

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A , B , and C are called the *lines* (as opposed to *phases*). In the delta system, it is readily shown that, in phasor notation,

$$\begin{aligned} V_{AB} &= V_{ab} = V_m e^{j0} \\ V_{AB} &= V_{bc} = V_m e^{j-120^\circ} \\ V_{CA} &= V_{ca} = V_m e^{j+120^\circ} \end{aligned} \quad (2-88)$$

while in the wye system,

$$\begin{aligned} V_{AB} &= V_{ab} - V_{bc} = V_{ao} - V_{bo} = \sqrt{3} V_m e^{j30^\circ} \\ V_{BC} &= V_{bc} - V_{ca} = V_{bo} - V_{co} = \sqrt{3} V_m e^{j-90^\circ} \\ V_{CA} &= V_{ca} - V_{ab} = V_{co} - V_{ao} = \sqrt{3} V_m e^{j+150^\circ} \end{aligned} \quad (2-89)$$

Thus, it is seen that in a delta system the line voltages are equal to the phase voltages. In a wye system, the line voltages are increased in magnitude by $\sqrt{3}$ and are shifted in phase by 30° . In a similar fashion, it is readily shown that the line currents in a wye system are equal to the phase currents, while in a delta, the line currents are increased by $\sqrt{3}$ and are shifted in phase by 30° .

Balanced Loads. A balanced 3-phase load is a set of three equal impedances connected either in wye or delta. Equations (2-74) and (2-75) may be used to convert from one to the other if needed.

Power Delivery, Balanced System. The power delivered from a balanced 3-phase source to a balanced 3-phase load is given by

$$P_{av} = \sqrt{3} V_{\text{line}} I_{\text{line}} \cos \phi$$

where $\cos \phi$ is, as before, the power factor of the load.

Unbalanced System. A 3-phase system that has either a nonsymmetric load or sources that differ in magnitude or whose phase difference is other than 120° is said to be unbalanced. Such circuits may be analyzed by any conventional method for circuit analysis.

Power Measurement in 3-Phase Systems. The power delivered to a 3-phase load, whether the system is balanced or unbalanced, may be measured with two wattmeters connected as shown in Fig. 2-38.

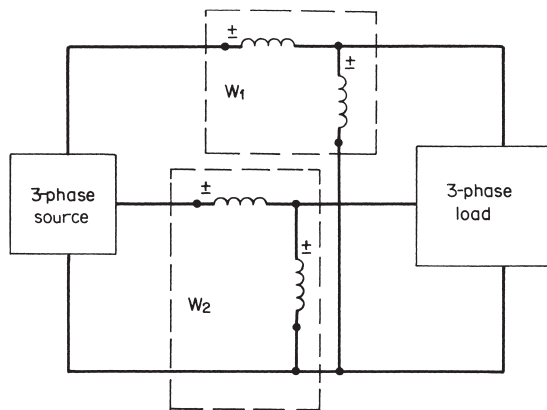


FIGURE 2-38 3-Phase system.

The total power is the sum of the readings of the two meters. If the power factors are small, one meter reading may be negative.

2.1.17 Symmetric Components

Resolution of an Unbalanced 3-Phase System into Balanced Systems. Let the three cube roots of unity, $1, e^{j(2\pi/3)}, e^{j(4\pi/3)}$, be $1, a, a^2$, where $j = \sqrt{-1}$,

$$a = 1 < 120^\circ = -0.5 + j0.866$$

and

$$a = 1 < 120^\circ = -0.5 - j0.866$$

Any three vectors, Q_a, Q_b, Q_c (which may be unsymmetric or unbalanced, that is, with unequal magnitudes or with phase differences not equal to 120°) can be resolved into a system of three equal vectors, $Q_{a_0}, Q_{a_1}, Q_{a_2}$ and two symmetrical (balanced) 3-phase systems $Q_{a_0}, a^2 Q_{a_1}, Q_{a_2}$ and $Q_{a_0}, a Q_{a_1}, Q_{a_2}$, the first of which is of positive-phase sequence and the second of negative-phase sequence. Thus

$$\begin{aligned} Q_a &= Q_{a_0} + Q_{a_1} + Q_{a_2} \\ Q_b &= Q_{a_0} + a^2 Q_{a_1} + a Q_{a_2} \\ Q_c &= Q_{a_0} + a Q_{a_1} + a^2 Q_{a_2} \end{aligned} \quad (2-90)$$

The values of the component vectors are

$$\begin{aligned} Q_{a_0} &= 1/3(Q_a + Q_b + Q_c) \\ Q_b &= 1/3(Q_a + a Q_b + a^2 Q_c) \\ Q_c &= 1/3(Q_a + a^2 Q_b + a Q_c) \end{aligned} \quad (2-91)$$

The three equal vectors Q_{a_0} are sometimes called the *residual quantities* or the *zero-phase*, or *uniphase, sequence system*. Any of the vectors Q_a, Q_b , or Q_c may have the value zero. If two of them are zero, the single-phase system may be resolved into balanced 3-phase systems by the preceding equations. The symbol Q may denote any vector quantity such as voltage, current, or electric charge.

There are similar relations for n -phase systems. See "Method of Symmetrical Coordinates Applied to the Solution of Polyphase Networks," by C. L. Fortescue, *Trans. AIEE*, 1918, p. 1027.

Short-Circuit Currents. The calculation of short-circuit currents in 3-phase power networks is a common application of the method of symmetrical components. The location of a probable short circuit of fault having been selected, three networks are computed in detail from the neutrals to the fault, one for positive, one for negative, and one for zero-phase sequence currents. The three phases are assumed to be identical, in ohms and in mutual effects, except in the connection of the fault itself. Let Z_1, Z_2 , and Z_0 be the ohms per phase between the neutrals and the fault in each of the networks, including the impedance of the generators.

Then, for a line-to-ground fault

$$I_{a_1} = I_{a_2} = I_{a_3} = \frac{I_c}{3} = \frac{V_a}{Z_1 + Z_2 + Z_0} \quad (2-92)$$

where V_a is the line-to-neutral voltage and I_{a_1} is the positive-phase-sequence current flowing to the fault in phase a and similarly for I_{a_2} and I_{a_0} . I_a is the total current flowing to the fault in phase a .

The component currents in phases b and c are derived from those in phase a , by means of the relations $I_{b_1} = a^2 I_{a_1}$, $I_{b_2} = a I_{a_2}$, $I_{b_0} = I_{a_0}$, $I_{c_1} = a I_{a_1}$, $I_{c_2} = a^2 I_{a_2}$, $I_{c_0} = I_{a_0}$. Each of the component currents divides in the branches of its own network according to the impedance of that network. Thus, each of the component currents, and therefore, the total current, at any part of the power system can be determined.

For a line-to-line fault between phases b and c

$$I_{a_1} = -I_{a_2} = \frac{V_a}{Z_1 + Z_2} \quad (2-93)$$

and

$$I_{a_0} = 0 \quad (2-94)$$

For a double line-to-ground fault between phases b and c and ground

$$I_{a_1} = \frac{V_a}{Z_1 + \frac{Z_2 Z_0}{Z_2 + Z_0}}$$

$$I_{a_0} = \frac{I_{a_1} Z_2}{Z_2 + Z_0}$$

and

$$I_{a_2} = -I_{a_1} - I_{a_0} \quad (2-95)$$

If there is no current in the power system before the fault occurs, the voltage V_a of every generator is the same in magnitude and phase. Such a condition often is assumed in calculated circuit-breaker duty and relay currents, although the effects of loads on the system can be included in the analysis.

In calculating power-system stability, however, it must be assumed that current exists in the lines before the fault occurs. The voltage V_a becomes the positive-sequence voltage at the point of fault before the fault occurs. A practical method of computing the positive-sequence current under fault conditions is to leave the positive-sequence network unchanged, with each generator at its own voltage and phase angle. The equivalent Z of the network need not be computed. Certain 3-phase impedances are inserted between line and neutral at the location of the fault. For a single line-to-line ground fault, $Z_2 + Z_0$ is inserted; for a line-to-line fault, Z_2 is inserted; and for a double line-to-ground fault, $Z_2 Z_0 / (Z_2 + Z_0)$ is inserted. This gives one phase of an equivalent balanced 3-phase circuit for which the positive-sequence currents driven by all the generators in all the branches under fault conditions can be found by means of a network analyzer or computed on a digital computer. The power transmitted after the fault occurs can be determined from these positive-sequence currents.

If it is desired to find the negative-sequence and zero-sequence currents (some relays are operated by the latter), they can be computed from Eqs. (2-92) to (2-95) that do not involve V_a , after finding I_{a_1} to the fault.

The impedance Z_f of each arc is mainly resistance. It may be brought into the computation. For single line-to-ground and double line-to-ground faults, Z_f is added to each of Z_1 , Z_2 , and Z_0 . For line-to-line faults, Z_f is added to Z_2 only.

Load Studies. In calculations relating to the steady-state operation of power systems, in which it is desired to determine the voltage, power, reactive power, etc. at various points, the loads may be designated by kilowatts and kilovars rather than by impedances. The effect of the impedance of the transmission and distribution lines, transformers, etc. of the network can be computed. The modern method is a process of iterations using a digital computer for the calculations. The division of current in branches, the voltage at various points, and the required ratings of synchronous capacitors can be determined.

Conditions can be estimated at one or two points and a solution for the rest of the network can be calculated on the basis of these assumptions. If the assumptions are not correct, discrepancies will appear at the end of the work. For instance, two different voltages may be obtained for the same point, one calculated before and one after going around a loop of the network. The necessary correction to the first estimates may be based on the discrepancies, thus giving successive approximations which are improvements on the preceding ones.

2.1.18 Additional 3-Phase Topics

Voltage Drop in Unsymmetrical Circuits. The voltage drop due to resistance, self-inductance, and mutual inductance in any conductor of a group of long, parallel, round, nonmagnetic conductors forming a single-phase or polyphase circuit, and with one or more conductors connected electrically in parallel, may be calculated by summing the flux due to each conductor up to a certain large distance u . The vectorial sum of all the currents is zero in a complete system of currents in the steady state, and the quantity u cancels out, so the result is the same no matter how large u may be. The currents may be unbalanced, and in addition, the arrangement of the conductors may be unsymmetrical.

The voltage drop in any conductor a of a group of round conductors $a, b, c, \dots$ is

$$I_a R_a = j0.2794(I_a \log_{10} S_a + I_b \log_{10} S_{ab} + I_c \log_{10} S_{ac} + \dots) \quad \text{V/mi at 60 Hz}$$

where $I_a + I_b + I_c = 0$, the values of the currents being complex quantities; R_a is resistance of conductor a , per mile; G_a is self-geometric mean distance of conductor a ; G_a is axial spacing between conductors a and b , etc. The values of G and S should be in the same units.

Armature Windings. The armature winding of a 3-phase generator or motor is an important type of electric circuit. Windings consisting of diamond-shaped coils, with two coil sides per slot, are connected in groups of coils, three groups or phase belts being opposite each pole. In general, the number of slots per pole per phase is a fraction equal to the average number of coils per phase belt. There are a larger number and a smaller number of coils per phase belt, differing by 1. The winding is usually found to be divided into repeatable sections of several poles each, the sections being duplicates of each other.

The number of poles in a section is found by writing the fraction equal to the number of slots divided by the number of poles and canceling factors to the extent possible. The denominator is the number of poles per section, and the numerator is the number of slots per section.

If the final value of the numerator is not divisible by 3, a balanced 3-phase winding cannot be made, since the windings for phases a, b , and c in a section each require the same number of slots, and they must be duplicates except for the phase shift of 120° . This gives rise to the rule for balanced 3-phase windings that the factor 3 must occur at least one more time in the number of slots than in the number of poles.

It can be shown that the slots of a repeatable section have phase angles which, when suitably drawn, are all different and equidistant. They fill the space of 180 electrical degrees like the blades of a Japanese fan. The angle between the vectors in this fan is

$$\beta = \frac{180}{\text{slots per section}} \quad \text{deg} \quad (2-96)$$

The vectors lying from 0° to 59° may be assigned to phase a or $-a$, those from 60° to 119° to phase $-c$ or c , and those from 120° to 179° to phase b or $-b$.

The phase angles for the upper coil sides of the slots should be tabulated to indicate the proper connections of the winding. Since the diamond coils are all alike, the total resulting voltage developed in the lower coil sides of a phase is a duplicate of that developed in the upper coil sides and can be added on by means of the pitch factor.

The phase angle between two adjacent slots is

$$\frac{\text{Poles per section} \times 180}{\text{Slots per section}} = q\beta \quad \text{deg} \quad (2-97)$$

2-34 SECTION TWO

where q is the number of poles in the repeatable section. From this, the phase angle for every slot in the section can be written. To save numerical work, especially where β is a fractional number of degrees, the angles may be expressed in terms of the angle β , as given in the example below. They may be expressed in degrees and fractions of a degree, but decimal values of degrees should not be used in this part of the work. The required accuracy is obtained by using fractions instead of decimals. Appropriate multiples of 180° should be subtracted to keep the angles less than 180° , thus indicating the relative position of each coil side with respect to the nearest pole. When an odd number times 180° has been subtracted, the coil side is tabulated as $-a$ instead of a , etc., since it will be opposite a south pole when a is opposite a north pole. The terminals of a coil marked $-a$ are reversed with respect to the terminals of a coil marked a with which it is in series.

Example 21 slots per repeatable section; 5 poles per section; $11/5$ slots per pole per phase.

$$\beta = \frac{180}{21} = 8\frac{4}{7} \quad \text{deg} \quad [\text{by Eq. 2-96}]$$

It is more convenient in this case to express the angles in terms of β rather than by fractions of degrees. Note that $21\beta = 180^\circ$ and $7\beta = 60^\circ$. The range for coils to be marked $\pm a$ is from 0 to 6β , inclusive; coils marked $\pm c$ from 7β to 13β ; and coils marked $\pm c$ from 14β to 20β . Subtract multiples of $21\beta = 180^\circ$. The angle between two adjacent slots is $q\beta = 5\beta$.

Tabulation of Phase Angles

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	[22]
0	5β	10β	15β	20β	$(25\beta)4\beta$	9β	14β	19β	$(24\beta)3\beta$	8β	13β	18β	$(23\beta)2\beta$	7β	12β	17β	$(22\beta)\beta$	6β	11β	16β	$[(21\beta)0]$
a	a	-c	b	b	-a	c	-b	-b	a	-c	-c	b	-a	c	c	-b	a	a	-c	b	[-a]

The seven vectors of phase a make a regular fan covering $7\beta = 60^\circ$.

The resulting terminal voltage produced by the coils of phase a is equal to the numerical sum of the voltages in those coils multiplied by the "distribution factor"

$$\frac{\sin(n\beta/2)}{n\sin(\beta/2)} \quad (2-98)$$

where n is the number of vectors in the regular fan covering 60° and β is the angle between adjacent vectors, given by Eq. (2-95). The number n is large, and the perimeter approaches the arc of a circle. Equation (2-97) is of the same form as the formula for breadth factor, which also is based on a vector diagram that is a regular fan.

The distribution factor for the winding of the foregoing example is

$$\frac{\sin \frac{7 \times 60^\circ}{2 \times 7}}{7 \sin \frac{60^\circ}{2 \times 7}} = \frac{\sin 30^\circ}{7 \sin 4\frac{2^\circ}{7}} = \frac{0.5}{7 \times 0.0746} = 0.956$$

Other possible balanced 3-phase windings for this example could be specified by having some of the vectors of phase a lie outside the 60° range. This would result in a lower distribution factor. The voltage, and hence the rating of the machine, would be lower by 2% or more than in the case described. The canceling of the harmonic voltages would apparently not be improved, and there would be no advantage to compensate for the reduction in kVA rating, which would correspond to a loss or waste of 2% or more of the cost of the machine.

2.1.19 Two Ports

Two Ports. A common use of electric circuits is to connect a source of electric energy or information to a load, often in such a way as to modify the signal in a prescribed way. In its basic form, such a circuit has one pair of input terminals and one pair of output terminals. Each of the four

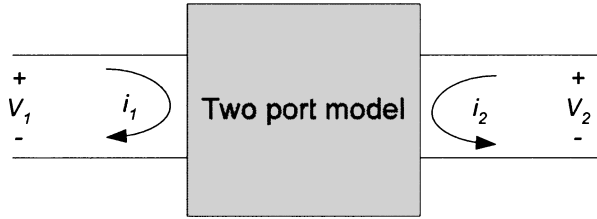


FIGURE 2-39 2-Port model with voltage and current definitions.

wires will have a current flow in or out of the circuit. Between any pair of terminals there will be a voltage.

If the structure of the circuit is such that the current flow into one terminal is equal to the current flow out of a second terminal, then that terminal pair is called a *port*. The circuit of Fig. 2-39 has two such pairs and is called a *2-port*. The circuit variables can be completely characterized at the terminals by the two currents and two voltages indicated in Fig. 2-39.

The last part of this section, “Filters,” is devoted to a special type of 2-port called a *filter*. In the section, a variety of relations among the terminal voltages and currents will be discussed. Six such sets are found to be useful. They are known as the open-circuit impedance parameters, short-circuit admittance parameters, hybrid parameters (two types), and transmission-line parameters (two types).

Open-Circuit Impedance Parameters. If the two currents are considered to be independent variables and the two voltages are dependent variables, then this pair of equations can be written as

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \quad [v] = [z][i] \quad (2-99)$$

The four numbers or functions in the square matrix characterize the network. They may be computed by any conventional method of circuit analysis and often are computed directly by computer-based software. They also may be measured. For example, the term z_{21} can be computed or measured as the ratio v_2/i_1 when i_2 is set to zero if being computed or made zero by an appropriate open circuit when measurements are being taken.

Short-Circuit Admittance Parameters. If the two voltages are dependent variables and the two currents independent, then these equations can be written as

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad [i] = [y][v] \quad (2-100)$$

Measurements and computations follow principles similar to those of open-circuit impedance parameters.

Hybrid Parameters. Voltages and currents may be mixed in their roles as independent and dependent variables in two ways, as indicated:

$$\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ v_2 \end{bmatrix} \quad (h \text{ parameters}) \quad (2-101)$$

$$\begin{bmatrix} i_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ i_2 \end{bmatrix} \quad (g \text{ parameters}) \quad (2-102)$$

Transmission-Line Parameters. The voltage and current at the input port may be used as independent variables with the output quantities as dependent variables, or the roles may be reversed.

TABLE 2-1 2-Port Conversions

$[z] = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$	$= \begin{bmatrix} \frac{y_{22}}{ y } & -\frac{y_{12}}{ y } \\ -\frac{y_{21}}{ y } & \frac{y_{11}}{ y } \end{bmatrix}$	$= \begin{bmatrix} \frac{ h }{h_{22}} & \frac{h_{12}}{h_{22}} \\ -\frac{h_{21}}{h_{22}} & \frac{1}{h_{22}} \end{bmatrix}$	$= \begin{bmatrix} \frac{1}{g_{11}} & -\frac{g_{12}}{g_{11}} \\ \frac{g_{21}}{g_{11}} & \frac{ g }{g_{11}} \end{bmatrix}$	$= \begin{bmatrix} \frac{A}{C} & \frac{ TL }{C} \\ \frac{1}{C} & \frac{D}{\gamma} \end{bmatrix}$	$= \begin{bmatrix} \frac{\delta}{\gamma} & \frac{1}{\gamma} \\ \frac{ TLI }{\gamma} & \frac{\alpha}{\gamma} \end{bmatrix}$
$[y] = \begin{bmatrix} \frac{z_{22}}{ z } & -\frac{z_{12}}{ z } \\ -\frac{z_{21}}{ z } & \frac{z_{11}}{ z } \end{bmatrix}$	$= \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$	$= \begin{bmatrix} \frac{1}{h_{11}} & -\frac{h_{12}}{h_{11}} \\ \frac{h_{21}}{h_{11}} & \frac{ h }{h_{11}} \end{bmatrix}$	$= \begin{bmatrix} \frac{ g }{g_{22}} & -\frac{g_{12}}{g_{22}} \\ -\frac{g_{21}}{g_{22}} & \frac{1}{g_{22}} \end{bmatrix}$	$= \begin{bmatrix} \frac{D}{B} & \frac{ TL }{B} \\ -\frac{1}{B} & \frac{A}{B} \end{bmatrix}$	$= \begin{bmatrix} \frac{\alpha}{\beta} & -\frac{1}{\beta} \\ \frac{ TLI }{\beta} & \frac{\delta}{\beta} \end{bmatrix}$
$[h] = \begin{bmatrix} \frac{ z }{z_{22}} & \frac{z_{12}}{z_{22}} \\ -\frac{z_{21}}{z_{22}} & \frac{1}{z_{22}} \end{bmatrix}$	$= \begin{bmatrix} \frac{1}{y_{11}} & -\frac{y_{12}}{y_{11}} \\ \frac{y_{21}}{y_{11}} & \frac{ y }{y_{11}} \end{bmatrix}$	$= \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$	$= \begin{bmatrix} \frac{g_{22}}{ g } & -\frac{g_{12}}{ g } \\ -\frac{g_{21}}{ g } & \frac{g_{11}}{ g } \end{bmatrix}$	$= \begin{bmatrix} \frac{B}{D} & \frac{ TL }{D} \\ -\frac{1}{D} & \frac{C}{D} \end{bmatrix}$	$= \begin{bmatrix} \frac{\beta}{\alpha} & -\frac{1}{\alpha} \\ -\frac{ TLI }{\alpha} & \frac{\gamma}{\alpha} \end{bmatrix}$
$[g] = \begin{bmatrix} \frac{1}{z_{11}} & -\frac{z_{12}}{z_{11}} \\ \frac{z_{21}}{z_{11}} & \frac{ z }{z_{11}} \end{bmatrix}$	$= \begin{bmatrix} \frac{ y }{y_{22}} & \frac{y_{12}}{y_{22}} \\ -\frac{y_{21}}{y_{22}} & \frac{1}{y_{22}} \end{bmatrix}$	$= \begin{bmatrix} \frac{h_{22}}{ h } & -\frac{h_{12}}{ h } \\ -\frac{h_{21}}{ h } & \frac{h_{11}}{ h } \end{bmatrix}$	$= \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$	$= \begin{bmatrix} \frac{C}{A} & \frac{ TL }{A} \\ \frac{1}{A} & \frac{B}{A} \end{bmatrix}$	$= \begin{bmatrix} \frac{\gamma}{\delta} & -\frac{1}{\delta} \\ \frac{ TLI }{\delta} & \frac{\beta}{\delta} \end{bmatrix}$
$[TL] = \begin{bmatrix} \frac{z_{11}}{z_{21}} & \frac{ z }{z_{21}} \\ \frac{1}{z_{21}} & \frac{z_{22}}{z_{21}} \end{bmatrix}$	$= \begin{bmatrix} -\frac{y_{22}}{y_{21}} & -\frac{1}{y_{21}} \\ -\frac{ y }{y_{21}} & -\frac{y_{11}}{y_{21}} \end{bmatrix}$	$= \begin{bmatrix} -\frac{ h }{h_{21}} & -\frac{h_{11}}{h_{21}} \\ -\frac{h_{22}}{h_{21}} & -\frac{1}{h_{21}} \end{bmatrix}$	$= \begin{bmatrix} \frac{1}{g_{21}} & \frac{g_{22}}{g_{21}} \\ \frac{g_{21}}{g_{21}} & \frac{ g }{g_{21}} \end{bmatrix}$	$= \begin{bmatrix} A & B \\ C & D \end{bmatrix}$	$= \begin{bmatrix} \frac{\delta}{ TLI } & \frac{\beta}{ TLI } \\ \frac{\gamma}{ TLI } & \frac{\alpha}{ TLI } \end{bmatrix}$
$[TLI] = \begin{bmatrix} \frac{z_{22}}{z_{12}} & \frac{ z }{z_{12}} \\ \frac{1}{z_{12}} & \frac{z_{11}}{z_{12}} \end{bmatrix}$	$= \begin{bmatrix} -\frac{y_{11}}{y_{12}} & -\frac{1}{y_{12}} \\ \frac{ y }{y_{12}} & -\frac{y_{22}}{y_{12}} \end{bmatrix}$	$= \begin{bmatrix} -\frac{1}{h_{12}} & -\frac{h_{11}}{h_{12}} \\ \frac{h_{22}}{h_{12}} & -\frac{ h }{h_{12}} \end{bmatrix}$	$= \begin{bmatrix} -\frac{ g }{g_{12}} & -\frac{g_{22}}{g_{12}} \\ -\frac{g_{11}}{g_{12}} & -\frac{1}{g_{12}} \end{bmatrix}$	$= \begin{bmatrix} \frac{D}{ TL } & \frac{B}{ TL } \\ \frac{C}{ TL } & \frac{A}{ TL } \end{bmatrix}$	$= \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$

$$\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} v_2 \\ i_2 \end{bmatrix} \quad (h \text{ parameters}) \quad (2-103)$$

$$\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} v_1 \\ i_1 \end{bmatrix} \quad (g \text{ parameters}) \quad (2-104)$$

2-Port Parameter Conversions. Any set of 2-port parameters may be converted to any other set through the use of Table 2-1. In this framework,

$$|y| = \det[y] = \det \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = y_{11}y_{22} - y_{12}y_{21} \quad (2-105)$$

With comparable interpretations for the other sets.

Equivalent Circuits for Two Ports. Equivalent circuits may be derived for any set of 2-port parameters. The process is shown for the $[h]$ parameters, but the technique is quite similar for the other combinations. Figure 2-40a shows the equivalent circuit.

Two-Port Analysis. From the equivalent circuit for a 2-port, circuit analysis is possible. Figure 2-40b shows a 2-port, with hybrid parameters, with a voltage source and a load resistor. Application of Kirchhoff's laws shows that

$$\frac{V_{\text{load}}}{V_{\text{source}}} = -\frac{R_L}{(R_{\text{source}} + h_{11})(1 + h_{22}R_{\text{load}}) + h_{12}R_{\text{load}}} \quad (2-106)$$

Similar analyses can be performed for any configuration that may arise.

Transmission Lines. The two sets of parameters denoted $[TL]$ and $[TLI]$ are called *transmission-line parameters* and are frequently computed or measured for power and communication transmission lines. Analysis with these parameters is substantially the same as that in the section on short-circuit admittance parameters above, though if the length of the line is more than about 10% of the wavelength involved, it is more convenient to use parameters based on standing-wave theory.

2.1.20 Transient Analysis and Laplace Transforms

Most transient analysis today is done with Laplace transform techniques, which provide the analyst a powerful method for finding both steady-state and transient computations simultaneously. It is necessary, however, to know initial conditions, the energy stored in capacitors and inductors.

Definition of a Laplace Transform. If a function of time $f(t)$ is known and defined for $t \geq 0$, then the (single-sided) Laplace transform is given by

$$Lf(t) = F(s) = \int_{0^-}^{\infty} f(t)e^{-st} dt \quad (2-107)$$

where s is a complex variable, $s = \sigma + j\omega$, chosen so that the integral will converge. In turn, σ is the real part of the variable, and ω is the imaginary part, but it becomes the frequency of sinusoidal functions measured in rad/s.

Laplace Transform Theorems. If the function $f(t)$ has the Laplace transform $F(s)$, then the theorems of Table 2-2 apply. In these theorems, the term $f(0^-)$ represents the initial condition, or the value of f at $t = 0$.

Laplace Transform Pairs. Table 2-3 presents a listing of the most common time functions and their Laplace transforms. These are sufficient for much of the analysis that is necessary. References include more extensive tables.

Initial Conditions. If a circuit has initial energy stored in it, that is, if any of the capacitors is charged or if any of the inductors has a nonzero current, then these conditions must be determined before a complete analysis can be done. In general, this will require knowledge of the circuit just before the circuit to be analyzed is connected. Normal circuit analysis methods can be used to determine the initial conditions.

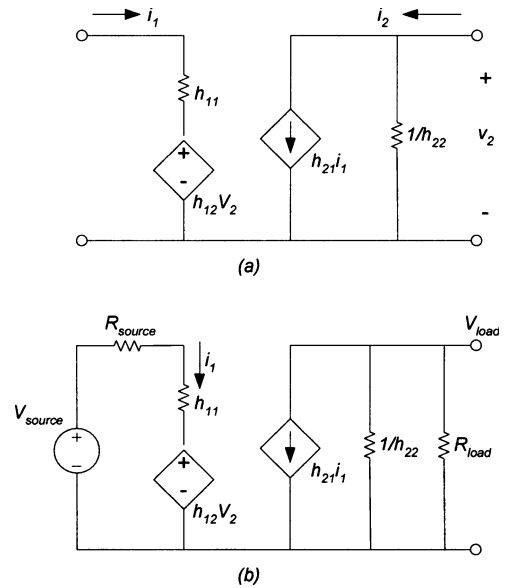


FIGURE 2-40 Transmission lines.

TABLE 2-2 Laplace Transform Theorems

Operation	Theorem
Derivative	$\mathcal{L} \frac{df(t)}{dt} = sF(s) - f(0^-)$
n th-order derivative	$\mathcal{L} \frac{d^n f}{dt^n} = s^n F(s) - s^{n-1}f(0^-) - \dots - f^{(n-1)}(0^-)$
Integral	$\mathcal{L} \int_0^t f(x)dx = \frac{F(s)}{s}$
Time shift	$\mathcal{L}[f(t - t_0) u(t - t_0)] = e^{-st_0} F(s)$ where u is the unit step function
Frequency shift	$\mathcal{L} e^{-at} f(t) = F(s + a)$
Frequency scaling	$\mathcal{L} f(at) = \frac{1}{a} F\left(\frac{s}{a}\right), a > 0$
Initial value	$\lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s)$ provided the limit exists $\lim_{t \rightarrow 0^+} f(t) = \lim_{s \rightarrow \infty} sF(s)$ provided the limit exists
Constant multiplier	$\mathcal{L} k f(t) = k \mathcal{L} f(t)$
Addition	$\mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1 F_1(s) + a_2 F_2(s)$ a_1, a_2 are constants

Transfer Functions. The ratio of the Laplace transform of a response function to the Laplace transform of an excitation function, when initial conditions are zero, is called a *transfer function*. Any or all of the elements of the 2-port parameter matrices can be a transfer function in addition to being numerics. If the substitution $s = j\omega$ is made in a transfer function, then the new function is a function of (sinusoidal) signals of varying frequency. The frequency is measured in rad/s.

Example of Laplace Analysis. Figure 2-41 shows a 2-node circuit with an initial charge on one of the capacitors and the Laplace domain equivalent circuit. Equations (2-108) to (2-112) show Kirchhoff's current law equations for the circuit, a Laplace domain solution for one of the voltages, a partial fraction expansion of the solution, and finally, the inverse transform. (To reduce arithmetic complexity, the network is scaled.)

$$\frac{2s}{s^2 + 4} = V_1(s) \left(1 + \frac{1}{2s} + s\right) - V_2(s) \left(\frac{1}{2s}\right) \quad (2-108)$$

$$0 = -V_2(s) \left(\frac{1}{2s}\right) + V_1(s) \left(1 + \frac{1}{2s} + s\right) - 4 \quad (2-109)$$

$$V_2(s) = \frac{4s^4 + 4s^3 + 18s^2 + 17s + 8}{(s^2 + 4)(s + 1)(s^2 + s + 1)} \quad (2-110)$$

TABLE 2-3 Laplace Transform Pairs

Name	Time function $f(t)$, $t > 0$	Laplace transform $F(s)$
Unit impulse	$\delta(t)$	1
Unit step	$u(t)$	$\frac{1}{s}$
Unit ramp	t	$\frac{1}{s^2}$
n th-order ramp	t^n	$\frac{n!}{s^{n+1}}$
Exponential	e^{-at}	$\frac{1}{s + a}$
Damped ramp	te^{-at}	$\frac{1}{(s + a)^2}$
Cosine	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
Sine	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
Damped cosine	$e^{-at} \cos \omega t$	$\frac{s + a}{(s + a)^2 + \omega^2}$
Damped sine	$e^{-at} \sin \omega t$	$\frac{\omega}{(s + a)^2 + \omega^2}$

$$V_2(s) = \frac{1.8}{s + 1} + \frac{-0.1077s - 0.1231}{s^2 + 4} + \frac{2.3077s + 0.23077}{s^2 + s + 1} \quad (2-111)$$

$$V_2(t) = 1.8e^{-t} - 0.1240 \cos(2t - 29.74^\circ) + 2.5420e^{-t/2} \cos(0.8660t + 24.79^\circ) \quad (2-112)$$

2.1.21 Fourier Analysis

Definition of Fourier Series. A periodic function $f(t)$ is defined as one that has the property

$$f(t) = f(t \pm nT) \quad (2-113)$$

where n is an integer and T is the period. If $f(t)$ satisfies the Dirichlet conditions, that is,

$f(t)$ has a finite number of finite discontinuities in the period T ,

$f(t)$ has a finite number of maxima and minima in the period T ,

the integral $\int_{t_0}^{t_0+T} |f(t)| dt$ exists. Then $f(t)$ can be written as a series of sinusoidal terms. Specifically,

$$f(t) = a_0 + \sum_{k=1}^{\infty} [a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)] \quad (2-114)$$

The coefficients may be found with these equations:

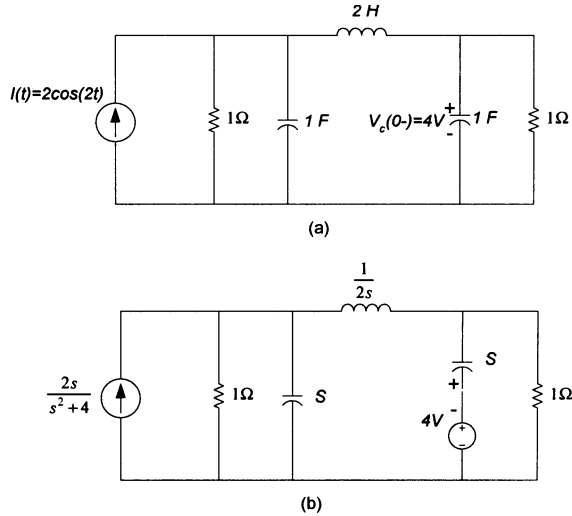


FIGURE 2-41 Example of Laplace transform circuit analysis: (a) 2-node circuit with one nonzero initial condition; (b) Laplace domain equivalent circuit equations and solution shown as Eqs. (2-108) to (2-112).

$$\begin{aligned}
 a_0 &= \frac{1}{T} \int_{t_0}^{t_0+T} f(t) dt \\
 a_k &= \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos(k\omega_0 t) dt \\
 b_k &= \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin(k\omega_0 t) dt
 \end{aligned} \tag{2-115}$$

Evaluation of Fourier Coefficients. If an analytic expression for $f(t)$ is known, then the integrals can be used to evaluate the coefficients, which are then a function of the integral variable k . If the function $f(t)$ is known numerically or graphically, then numerical integration is required. Such integration is readily done with suitable computer software.

Effect of Symmetry. If the function $f(t)$ is even, that is, $f(t) = f(-t)$, then the expressions for the coefficients become

$$\begin{aligned}
 a_0 &= \frac{2}{T} \int_0^{T/2} f(t) dt \\
 a_k &= \frac{4}{T} \int_0^{T/2} f(t) \cos(k\omega_0 t) dt \\
 b_k &= 0 \quad \text{for all } k
 \end{aligned} \tag{2-116}$$

If $f(t)$ is odd, that is $f(t) = -f(-t)$, then

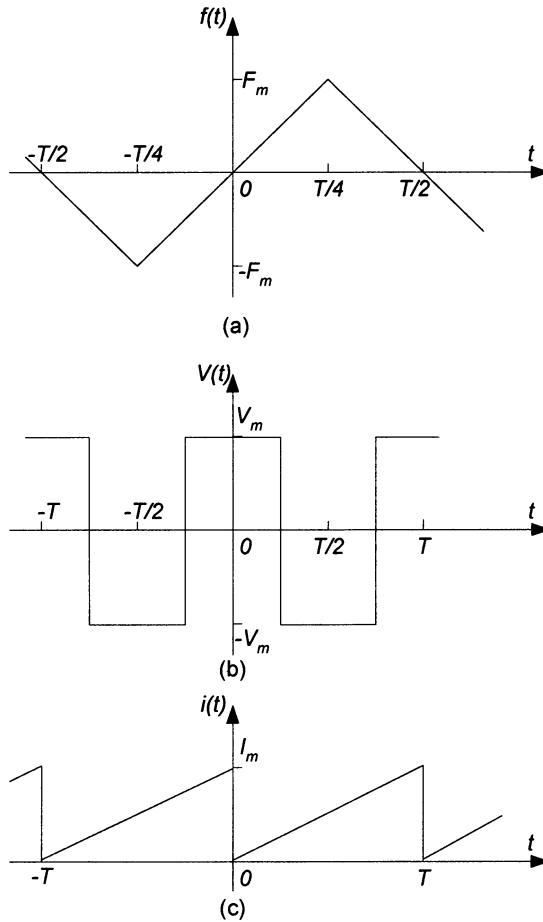


FIGURE 2-42 Waveforms for Fourier analyses [see Eqs. (2-118) to (2-120)]: (a) triangular wave, odd symmetry; (b) square wave, even symmetry; (c) ramp function.

$$a_k = 0 \quad \text{for} \quad k = 0, 1, 2, 3, \dots$$

$$b_k = \frac{4}{T} \int_0^{T/2} \sin(k\omega_0 t) dt \quad (2-117)$$

Fourier Series Examples. Figure 2-42 shows three waveforms: a triangular signal written as an odd function, a square wave written as an even function, and a ramp function. For these three signals, the Fourier series are

$$f(t) = \frac{8F_m}{\pi^2} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi}{2}\right) \sin(n\omega_0 t)$$

$$v(t) = \frac{4V_m}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right) \sin(n\omega_0 t) \quad (2-118)$$

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$$i(t) = \frac{I_m}{2} - \frac{I_m}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(n\omega_0 t)$$

Average Power Calculations. If the Fourier series given in Eq. (2-114) represents a current or voltage, then the effective value of this current or voltage is given by

$$F_{\text{eff}} = \sqrt{a_0^2 + \frac{1}{2} \sum_{k=1}^{\infty} (a_k^2 + b_k^2)} \quad (2-119)$$

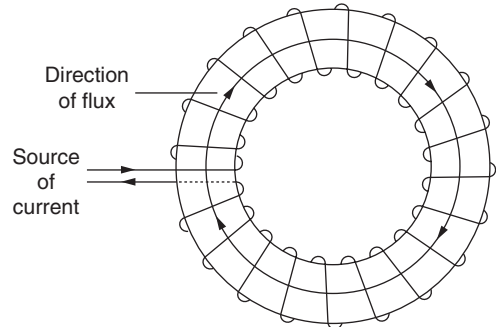


FIGURE 2-43 Closed magnetic circuit.

2.1.22 The Magnetic Circuit

The Simple Magnetic Circuit. A simple magnetic circuit is a uniformly wound torus ring (Fig. 2-43). The relation between the mmf F and the flux ϕ is similar to Ohm's law, namely,

$$F = R\phi \quad \text{At} \quad (2-120)$$

where R is called the *reluctance* of the magnetic circuit. The relation is sometimes written in the form

$$\phi = PF$$

where $p = 1/R$ is called the *permeance* of the magnetic circuit. Reluctance is analogous to resistance, and permeance is analogous to conductance of an electric circuit.

$$F = NI \quad \text{At} \quad (2-121)$$

where N is the number of turns of conductor around the magnetic circuit, as in Fig. 2-43, and I is the current in the conductor, in amperes.

Permeability and Reluctivity. The reluctance of a uniform magnetic path (Fig. 2-43) is proportional to its length l and inversely proportional to its cross section A .

$$R = \nu \frac{l}{A} \quad \text{At/Wb} \quad (2-122)$$

and

$$P = \mu \frac{A}{l} \quad \text{Wb/At} \quad (2-123)$$

In these expressions, ν is called the *reluctivity* and μ the *permeability* of the material of the magnetic path, it being assumed that there is no residual magnetism. The dimensions l and A are in metric units. For a vacuum, air, or other nonmagnetic substance, the reluctivity and permeability are usually written ν_0 and μ_0 , and their values are $1/(4\pi \times 10^{-7})$ and $4\pi \times 10^{-7}$, respectively.

Magnetic Field Intensity. Magnetic field intensity H is defined as the mmf per unit length of path of the magnetic flux. It is known also as the *magnetizing force* or the *magnetic potential gradient*. In a uniform field,

$$H = \frac{F}{l} \quad \text{At/m} \quad (2-124)$$

In a nonuniform magnetic circuit,

$$H = \frac{\partial F}{\partial l} \quad (2-125)$$

Inversely, for a uniform field,

$$F = HI \quad (2-126)$$

and for a nonuniform field,

$$F = \int Hdl \quad (2-127)$$

By Ampere's law, when this integral is taken around a complete magnetic circuit,

$$F = \oint Hdl = I \quad (2-128)$$

where I is the total current, in amperes, surrounded by the magnetic circuit. The circle on the integral sign indicates integration around the complete circuit. In Eqs. (2-126) to (2-128), it is presumed that H is directed along the length of l ; otherwise, the factor $\cos \theta$ must be added to the product Hdl , where θ is the angle between H and dl .

Flux Density. Flux density B is the magnetic flux per unit area, the area being perpendicular to the direction of the magnetic lines of force. In a uniform field,

$$B = \frac{\phi}{A} \quad \text{T (or Wb/m}^2\text{)} \quad (2-129)$$

Reluctances and Permeances in Series and in Parallel. Reluctances and permeances are added like resistances and conductances, respectively. That is, *reluctances are added when in series, and permeances are added when in parallel*. If several permeances are given connected in series, they are converted into reluctances by taking the reciprocal of each. If reluctances are given in a parallel combination, they are similarly converted into permeances.

Magnetization Characteristic or Saturation Curve. The magnetic properties of steel or iron are represented by a saturation or magnetization curve (Fig. 2-44). Magnetic field intensities H in ampere-turns per meter are plotted as abscissas and the corresponding flux densities B in teslas (webers per square meter) as ordinates.

The practical use of a magnetization curve may be best illustrated by an example. Let it be required to find the number of exciting ampere-turns for magnetizing a steel ring so as to produce in it a flux of 1.68 mWb. Let the cross section of the ring be 10.03 by 0.04 m and the mean diameter 0.46 m. Let the quality of the material be represented by the curve in Fig. 2-44. The flux density is $1.68 \times 10^{-3} / (0.03 \times 0.04) = 1.4 \text{ Wb/m}^2$. For this flux density, the corresponding abscissa from the curve is about 18 At/m. The total required number of ampere-turns is then $18 \times \pi \times 46 = 2600$.

For curves of various grades of steel and iron, see Sec. 4. The principal methods for experimentally obtaining magnetization curves will be found in Sec. 3.

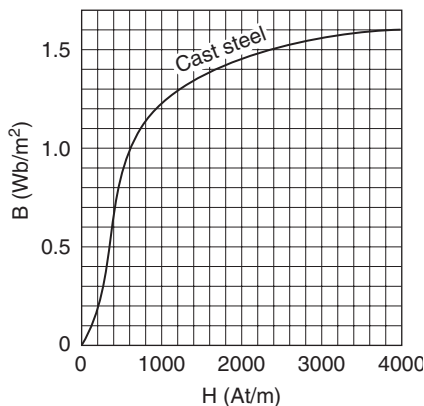


FIGURE 2-44 Typical BH curve.

Ampere-Turns for an Air Gap. In a magnetic circuit consisting of iron with one or more small air gaps in series with the iron, the magnetic flux density in each of the air gaps may be considered approximately uniform. If the length across a given air gap in the direction of the flux is l m, the ampere-turns required for that air gap is given by the equation

$$\text{At/m} = \frac{B(T)}{4\pi \times 10^{-7}} = 7.958 \times 10^5 B(T) \quad (2-130)$$

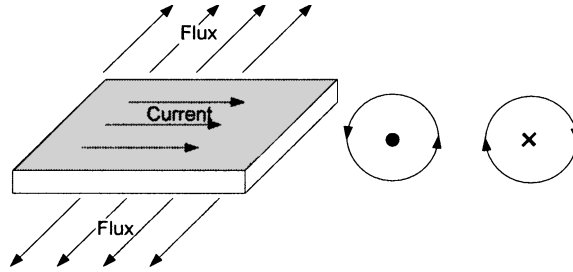


FIGURE 2-45 Relation between direction of current and flux.

The ampere-turns for each portion of iron, computed from iron magnetization curves such as Fig. 2-44, and the ampere-turns for the air gaps are added together to give the ampere-turns for the complete magnetic circuit.

Analysis of Magnetization Curve. Three parts are distinguished in a magnetization curve (Fig. 2-44): the lower, or nearly straight, part; the middle part, called the *knee* of the curve; and the upper part, which is nearly a straight line. As the magnetic intensity increases, the corresponding flux density increases more and more slowly, and the iron is said to approach saturation (see Sec. 4).

Magnetization per Unit Volume and Susceptibility. If a portion of ferromagnetic material is magnetized by an mmf, H At/m, the resulting flux density in teslas may be written as

$$B = \mu_0(H + M) \quad (2-131)$$

where M is the magnetization per unit volume of the material (see Sec. 4).

The ratio of M/H is symbolized by x and is called the *magnetic susceptibility*. It is the excess of the ratio of $B/\mu_0 H$ above unity, that is,

$$x = \frac{B}{\mu_0 H} - 1 \quad (2-132)$$

This is a dimensionless quantity. See Sec. 1.

The Right-Handed-Screw Rule. The direction of the flux produced by a given current is determined as shown in Fig. 2-45 (see also Fig. 2-43). If the current is established in the direction of rotation of a right-handed screw, the flux is in the direction of the progressive movement of the screw. If the current in a straight conductor is in the direction of the progressive motion of a right-handed screw, then the flux encircles this conductor in the direction in which the screw must be rotated in order to produce this motion. The dots in the figure indicate the direction of flux or current toward the reader, and the crosses away from the reader.

Fleming's Rules. The relative direction of flux, voltage, and motion in a revolving-armature generator may be determined with the right hand by placing the thumb, index, and middle fingers so as to form the three axes of a coordinate system and pointing the index finger in the direction of the flux (north to south) and the thumb in the direction of motion; the middle finger will give the direction of the generated voltage (Fig. 2-46). In the same way, in a revolving-armature motor, by using the left hand and pointing the index finger in the direction of the flux and the middle finger in the direction of the current in the armature conductor, the thumb will indicate the direction of

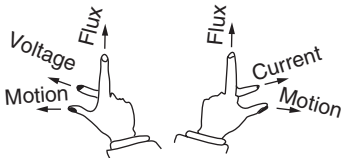


FIGURE 2-46 Fleming's generator and motor rules.

the force and, therefore, the resulting motion. These two rules, indicated in Fig. 2-46, are known as *Fleming's rules*.

Magnetic Tractive Force. The attracting force of a magnet is

$$F = \frac{1}{2} \frac{AB^2}{\mu_0} = \frac{AB^2}{8\pi \times 10^{-7}} \quad \text{newtons} \quad (2-133)$$

where B is the flux density in the air gap, expressed in teslas (webers per square meter), and A is the total area of the contact between the armature and the core, in square meters. The mass that can be supported is dependent on the gravity field in which the mass and magnet are located.

Magnetic Force, or Torque. The mechanical force, or the torque, between two parts of a magnetic or electric circuit may in some cases be conveniently calculated by making use of the principle of *virtual displacements*. An infinitesimal displacement between the two parts is assumed. The energy supplied from the source of current is then equal to the mechanical energy for producing the motion, plus the change in the stored magnetic energy, plus the energy for resistance loss.

When the differential motion ds m of a part of a circuit carrying a current I A changes its self-inductance by a differential dL H, the mechanical force on that part of the circuit, in the direction of the motion, is

$$F = \frac{1}{2} I^2 \frac{dL}{ds} \quad \text{newtons} \quad (2-134)$$

When the motion of one coil or circuit carrying a current I_1 A changes its mutual inductance by a differential dM H with respect to another coil or circuit carrying a current I_2 A, the mechanical force on each coil or circuit, in the direction of the motion, is

$$F = I_1 I_2 \frac{dM}{ds} \quad \text{newtons} \quad (2-135)$$

where ds represents the differential of distance in meters. For a discussion of self-inductance and mutual inductance L and M , see Sec. 2.1.6.

2.1.23 Hysteresis and Eddy Currents in Iron

The Hysteresis Loop. When a sample of iron or steel is subjected to an alternating magnetization, the relation between B and H is different for increasing and decreasing values of the magnetic intensity (Fig. 2-47). This phenomenon is due to irreversible processes which result in energy dissipation, producing heat. Each time the current wave completes a cycle, the magnetic flux wave also must complete a cycle, and the elementary magnets are turned. The curve $AefBcdA$ in Fig. 2-47 is called the *hysteresis loop*.

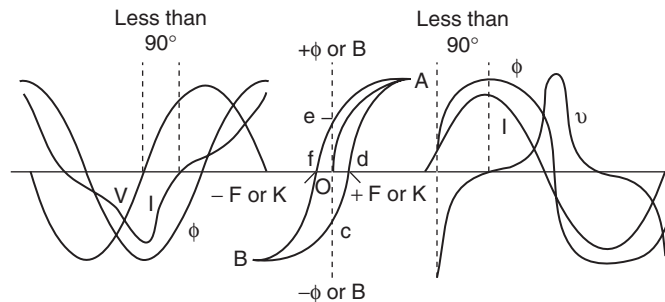


FIGURE 2-47 Periodic waves of current, flux, and voltage; hysteresis loop.

Retentivity. If the coil shown in Fig. 2-43 is excited with alternating current, the ampere-turns and consequently the mmf will, at any instant, be proportional to the instantaneous value of the exciting current. Plotting a B - H (or ϕ - F) curve (Fig. 2-47) for one cycle yields the closed loop $AefBcdA$. The first time the iron is magnetized, the *virgin*, or *neutral*, curve OA will be produced, but it cannot be produced in the reverse direction AO because when the mmf drops to zero there will always be some remaining magnetism ($+Oe$ or $-Oc$). This is called *residual magnetism*; to reduce this to zero, an mmf ($-Of$ or $+Od$) of opposite polarity must be applied. This mmf is called the *coercive force*.

Wave Distortion. In Fig. 2-47 the instantaneous values of the exciting current I (which is directly proportional to the mmf) and the corresponding values of the flux ϕ and voltage V (or v) are plotted against time as abscissas, beside the hysteresis loop. (a) If the voltage applied to the coil is *sinusoidal* (V , to the left), the current wave is distorted and displaced from the corresponding sinusoidal flux wave. The latter wave is in quadrature with the voltage wave. (b) If the *current* through the coil is *sinusoidal* (I , to the right), the flux is distorted into a flat-top wave and the induced voltage v is peaked.

Components of Exciting Current. The alternating current that flows in the exciting coil (Fig. 2-47) may be considered to consist of two components, one exciting magnetism in the iron and the other supplying the iron loss. For practical purposes, both components may be replaced by equivalent sine waves and phasors (Fig. 2-48) (see Sec. 2.1.11). We have

$$I_r = I \cos \theta = \text{power component of current}$$

$$P_h = IV \cos \theta = I_r V = \text{iron loss in watts} \quad (2-136)$$

$$I_m = I \sin \theta = \text{magnetizing current}$$

where I is the total exciting current, and θ the angle of time-phase displacement between current and voltage.

Hysteretic Angle. Without iron loss, the current I would be in phase quadrature with V . For this reason, the angle $\alpha = 90 - \theta$ is called the *angle of hysteretic advance of phase*.

$$\sin \alpha = \frac{I_r}{I} = \frac{I_r N}{IV} = \frac{W_{\text{loss}}}{VA} \quad (2-137)$$

In practice, the measured loss usually includes eddy currents, so the name *hysteretic* is somewhat of a misnomer.

The energy lost per cycle from hysteresis is proportional to the area of the hysteresis loop (Fig. 2-47). This is a consequence of the evaluation over a cycle of Eq. (2-13).

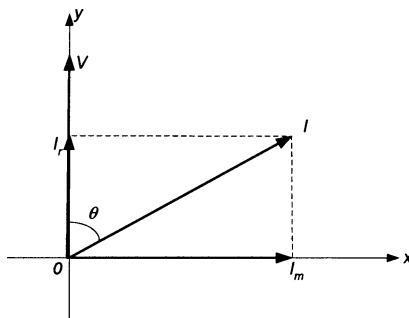


FIGURE 2-48 Components of exciting current; hysteretic angle.

Steinmetz's Formula. According to experiments by C.P. Steinmetz, the heat energy due to hysteresis released per cycle per unit volume of iron is approximately

$$W_h = \eta B_{\text{max}}^{1.6} \quad (2-138)$$

The exponent of B_{max} varies between 1.4 and 1.8 but is generally taken as 1.6. Values of the hysteresis coefficient η are given in Sec. 4.

Eddy-Current Losses. Eddy-current losses are $I^2 R$ losses due to secondary currents (Foucault currents) established in those parts of the circuit which are interlinked with alternating or pulsating flux. Refer to

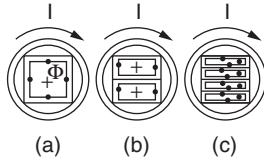


FIGURE 2-49 Section of a transformer core.

Fig. 2-49, which shows a cross section of a transformer core. The primary current I produces the alternating flux Φ , which by its change generates a voltage in the core; this voltage then sets up the secondary current i . Now, if the core is divided into two (b), four (c), or n parts, the voltage in each circuit is $v/2$, $v/4$, v/n , and the conductance is $g/2$, $g/4$, g/n , respectively. Thus, the loss per lamination will be $1/n^3$ times the loss in the solid core, and the total loss is $1/n^2$ times the loss in the solid core. When power is computed, the power is given by

$$P_h = \eta f \beta_{\max}^{1.6} \quad (2-139)$$

Eddy currents can be greatly reduced by laminating the circuit, that is, by making it up of thin sheets each electrically insulated from the others. The same purpose is accomplished by using separately insulated strands of conductors or bundles of wires.

A formula for the eddy loss in conductors of circular section, such as wire, is

$$P_e = \frac{(\pi r f B_{\max})^2}{4\rho} \quad \text{W/m}^3 \quad (2-140)$$

where r is the radius of the wire in meters, f is the frequency in hertz (cycles per second), $B_{\max}$ is the maximum flux density in teslas, and ρ is the specific resistance in ohm meters.

A formula for the loss in sheets is

$$P_e = \frac{(\pi r f B_{\max})^2}{6\rho} \quad \text{W/m}^3 \quad (2-141)$$

where t is the thickness in meters, f is the frequency in hertz (cycles per second), $B_{\max}$ is the maximum flux density in teslas, and ρ is the specific resistance in ohm meters. The specific resistance of various materials is given in Sec. 4.

Effective Resistance and Reactance. When an A.C. circuit has appreciable hysteresis, eddy currents, and skin effect, it can be replaced by a circuit of equivalent resistances and equivalent reactances in place of the actual ones. These effective quantities are so chosen that the energy relations are the same in the equivalent circuit as in the actual one. In a series circuit, let the true power lost in ohmic resistance, hysteresis, and eddy currents be P , and the reactive (wattless) volt-amperes, Q . Then the effective resistance and reactance are determined from the relations

$$i^2 r_{\text{eff}} = P \quad i^2 x_{\text{eff}} = Q \quad (2-142)$$

In a parallel circuit, with a given voltage, the equivalent conductances and susceptance are calculated from the relations

$$e^2 g_{\text{eff}} = P \quad e^2 b_{\text{eff}} = Q \quad (2-143)$$

Such equivalent electric quantities, which replace the core loss, are used in the analytic theory of transformers and induction motors.

Core Loss. In practical calculations of electrical machinery, the total core loss is of interest rather than the hysteresis and the eddy currents separately. For such computations, empirical curves are used, obtained from tests on various grades of steel and iron (see Sec. 4).

Separation of Hysteresis Losses from Eddy-Current Losses. For a given sample of laminations, the total core loss P , at a constant flux density and at variable frequency f , can be represented in the form

$$P = af + bf^2 \quad (2-144)$$

where af represents the hysteresis loss and bf^2 the eddy, or Foucault-current, loss, a and b being constants. The voltage waveform should be very close to a sine wave. If we write this equation for two known frequencies, two simultaneous equations are obtained from which a and b are determined.

It is convenient to divide the foregoing equation by f , because the form

$$\frac{P}{f} = a + bf \quad (2-145)$$

represents a straight line relating P/f and f . Known values of P/f are plotted against f as abscissas, and a straight line having the closest approximation to the points is drawn. The intersection of this line with the axis of ordinates gives a ; b is calculated from the preceding equation. The separate losses are calculated at any desired frequency from af and bf^2 , respectively.

2.1.24 Inductance Formulas

Inductance. The properties of self-inductance and mutual inductance are defined in Sec. 2.1.6. In these paragraphs, inductance relations for common geometries are given.

Torus Ring or Toroidal Coil of Rectangular Section with Nonmagnetic Core (Fig. 2-43). Inductance of a rectangular toroidal coil, uniformly wound with a single layer of fine wire, is

$$L = 2 \times 10^{-7} N^2 b \left(\ln \frac{r_2}{r_1} \right) \quad \text{henrys} \quad (2-146)$$

where N equals the number of turns of wire on the coil, b is the axial length of the coil in meters, and r_2 and r_1 are the outer and inner radial distances in meters.

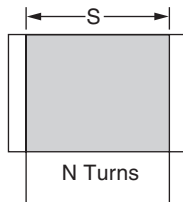
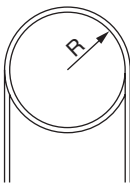
Torus Ring or Toroidal Coil of Circular Section with Nonmagnetic Core (Fig. 2-43). A toroidal coil of circular section, uniformly wound with a single layer of fine wire of N turns, has an inductance of

$$L = 4\pi \times 10^{-7} N^2 (g - \sqrt{g^2 - a^2}) \quad \text{henrys} \quad (2-147)$$

where g is the mean radius of the toroidal ring and a is the radius of the circular cross section of the core, both measured in meters.

Inductance of a Very Long Solenoid. A solenoid uniformly wound in a single layer of fine wire possesses an inductance of

$$L = \frac{4\pi^2 \times 10^{-7} N^2 R^2}{S} \quad \text{henrys} \quad (2-148)$$



where R and S are the radius and length of the solenoid in meters, as illustrated in Fig. 2-50. The assumption is made that S is very large with respect to R .

Inductance of the Finite Solenoid. The inductance of a short solenoid is less than that given by Eq. (2-148), by a factor k (a dimensionless quantity). The inductance relation then is

$$L = k \frac{4\pi^2 \times 10^{-7} N^2 R^2}{S} \quad \text{henrys} \quad (2-149)$$

where the values of k for various ratios of R and S are given in Fig. 2-51. Inductance relations for other configurations of coils are given by Boast (1964).

Inductance per Unit Length of a Coaxial Cable. For low-frequency applications, where skin effect is not predominant (uniform current density over nonmagnetic current-carrying cross sections), the inductance per unit length of a coaxial cable is

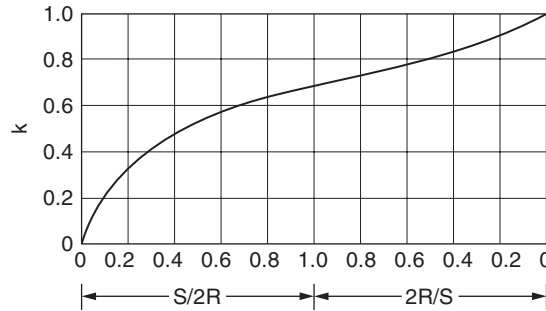


FIGURE 2-51 Factor k of Eq. (2-149). (From W. B. Boast, *Vector Fields*, New York, Harper & Row, 1964.)

$$L = l = \frac{10^{-7}}{2} \left[1 + 4 \ln \frac{R_2}{R_1} + \frac{4R_3^4}{(R_3^2 - R_2^2)^2} \ln \frac{R_3}{R_2} - \frac{3R_3^2 - R_2^2}{R_3^2 - R_2^2} \right] \quad \text{H/m} \quad (2-150)$$

where R_1 , R_2 , and R_3 are the radii of the inner conductor, the inner radius of the outer conductor, and the outer radius of the outer conductor, in meters, respectively, as shown in Fig. 2-52. For very thin outer shells, the last two terms drop out of the equation, and for very small inner conductors, the first term becomes less important. For high-frequency applications, the first, third, and fourth terms are all suppressed, and for the extreme situation where all the current is essentially at the boundaries formed by R_1 and R_2 , respectively, the inductance per unit length becomes

$$l = 2 \times 10^{-7} \ln(R_2/R_1) \quad \text{H/m} \quad (2-151)$$

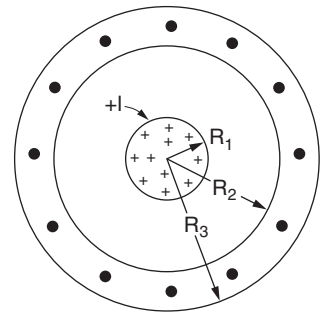


FIGURE 2-52 Coaxial cable.

Inductance of Two Long, Cylindrical Conductors, Parallel and External to Each Other. The inductance per unit length of two separate parallel conductors is

$$l = 10^{-7} \left(1 + 4 \ln \frac{D}{\sqrt{R_1 R_2}} \right) \quad \text{H/m} \quad (2-152)$$

where D is the distance between centers of the two cylinders and R_1 and R_2 are the radii of the conductor cross sections.

If $R_1 = R_2 = R$ and the skin-effect phenomenon applies as at very high frequencies, the inductance per unit length becomes

$$l = 4 \times 10^{-7} \left(1 + 4 \ln \frac{D}{\sqrt{R_1 R_2}} \right) \quad \text{H/m} \quad (2-153)$$

Inductance of Transmission Lines. The inductance relationships used in predicting the performance of power-transmission systems often involve the effects of stranded and bundled conductors operating in parallel, as well as configurations of these groups of current-carrying elements of one phase group of the system coordinated with similar groups constituting other phases, in polyphase systems. In such systems, the several current-carrying elements of a phase are considered mathematically

as a cylindrical shell of current of radius D_s (meters) called the *self-geometric mean radius* of the phase, and the mutual distances (between the current in a particular phase and the other [return] currents in the other phases) are replaced by a distance D_m (meters) called the *mutual geometric mean distance* to the return. The inductance of all phases may be balanced by transposing the conductors over the length of the transmission line so that each phase occupies all positions equally in the length of the line.

The inductance per phase is then one-half as large as that of Eq. (2.153), that is,

$$l = 2 \times 10^{-7} 4 \ln(D_m/D_s) \quad \text{H/m} \quad (2-154)$$

The references related to methods for computing the geometric mean distances D_m and D_s are available in Bibliography at the end of the section.

Leakage Inductance. In electrical apparatus, such as transformers, generators, and motors, in which the greater part of the flux is carried by an iron core, the difference between self-inductance and mutual inductance of the primary and secondary windings is small. This small difference is called *leakage inductance*. It is of great importance in the characteristics and operation of the apparatus and is usually calculated or measured separately. The loss in voltage in such apparatus, due to inductance, is associated with the leakage.

Magnetizing Current. The mutual inductance of the windings of apparatus with iron cores is not usually stated in henrys, but the effective alternating current required to produce the flux is stated in amperes and is called the *exciting current*. One component of this current supplies the energy corresponding to the core loss. The remaining component is called *magnetizing current*. Solenoids and other coils with only one winding are usually treated in a similar manner when they have iron cores. The exciting current usually does not have a sine-wave form. See Fig. 2-47.

2.1.25 Skin Effect

Real, or ohmic, resistance is the resistance offered by the conductor to the passage of electricity. Although the specific resistance is the same for either alternating or continuous current, the total resistance of a wire is greater for alternating than for continuous current. This is due to the fact that there are induced emfs in a conductor in which there is alternating flux. These emfs are greater at the center than at the circumference, so the potential difference tends to establish currents that oppose the current at the center and assist it at the circumference. The current is thus forced to the outside of the conductor, reducing the effective area of the conductor. This phenomenon is called *skin effect*.

Skin-Effect Resistance Ratio. The ratio of the A.C. resistance to the D.C. resistance is a function of the cross-sectional shape of the conductor and its magnetic and electrical properties as well as of the frequency. For cylindrical cross sections with presumed constant values of relative permeability μ_r and resistivity ρ , the function that determines the skin-effect ratio is

$$mr = \sqrt{\frac{8\pi^2 \times 10^{-7} f \mu_r}{\rho}} \quad (2-155)$$

where r is the radius of the conductor and f is the frequency of the alternating current. The ratio of R , the A.C. resistance, to R_0 , the D.C. resistance, is shown as a function of mr in Fig. 2.53.

Steel Wires and Cables. The skin effect of steel wires and cables cannot be calculated accurately by assuming a constant value of the permeability, which varies throughout a large range during every cycle. Therefore, curves of measured characteristics should be used. See *Electrical Transmission and Distribution Reference Book*, 4th ed., 1950.

Skin Effect of Tubular Conductors. Cables of large size are often made so as to be, in effect, round, tubular conductors. Their effective resistance due to skin effect may be taken from the curves of Sec. 4. The skin-effect ratio of square, tubular bus bars may be obtained from semiempirical formulas in the paper "A-C Resistance of Hollow, Square Conductors," by A. H. M. Arnold, *J. IEE (London)*, 1938,

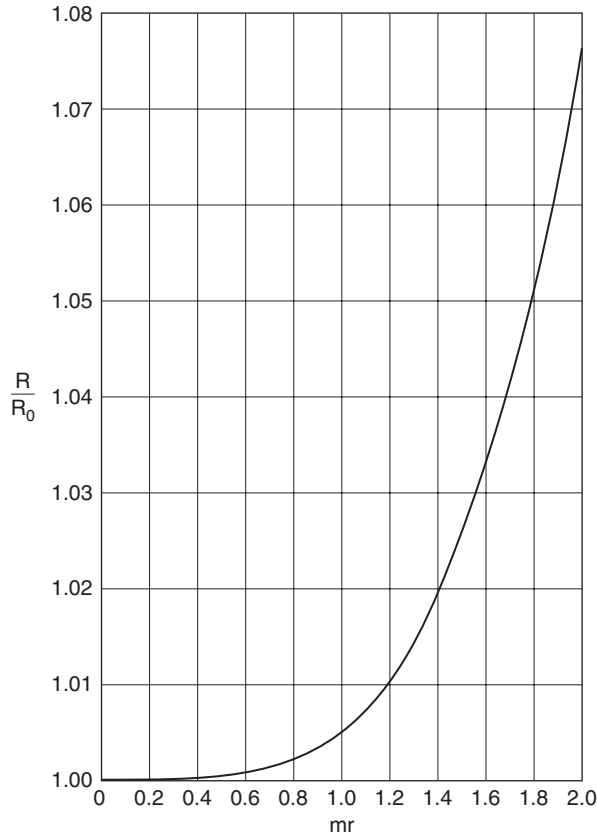


FIGURE 2-53 Ratio of A.C. to D.C. resistance of a cylindrical conductor. (From W. D. Stevenson, *Elements of Power Systems Analysis*, New York, McGraw-Hill; 1962.)

vol. 82, p. 537. These formulas have been compared with tests. The resistance ratio of square tubes is somewhat larger than that of round tubes. Values may be read from the curves of Fig. 4, Chap. 25, of *Electrical Coils and Conductors*.

Penetration Formula. For wires and tubes (and approximately for other compact shapes) where the resistance ratio is comparatively large, the conductor can be approximately considered to be replaced by its outer shell, of thickness equal to the “penetration depth,” given by

$$\delta = \frac{1}{2\pi} \sqrt{\frac{10^7 \rho}{f \mu_r}} \quad \text{meters} \quad (2-156)$$

where ρ is the resistivity in ohm-meters, and μ_r is a presumed constant value of relative permeability. The resistance of the shell is then the effective resistance of the conductor. For Eq. (2-156) to be applicable, δ should be small compared with the dimensions of the cross section. In the case of tubes, δ is, evidently, less than the thickness of the tube. See Eq. (30), Chap. 19, of *Electrical Coils and Conductors*.

2.1.26 Electrostatics

Electrostatic Force. Electrically charged bodies exert forces on one another according to the following principles:

1. Like charged bodies repel; unlike charged bodies attract one another.
2. The force is proportional to the product of the magnitudes of the charges on the bodies.
3. The force is inversely proportional to the square of the distance between charges if the material in which the charges are immersed is extensive and possesses the same uniform properties in all directions.
4. The force acts along the line joining the centers of the charges.

Two concentrated charges Q_1 and Q_2 coulombs located R m apart experience a force between them of

$$F = \frac{Q_1 Q_2}{4\pi e_0 R^2} \quad \text{newtons} \quad (2-157)$$

where $e_0 = 8.85419 \times 10^{-12}$ F/m and is the permittivity of free space.

Electrostatic Potential. The electric potential resulting from the location of charged bodies in the vicinity is called *electrostatic potential*. The potential at R m from a concentrated charge Q C is

$$\phi = \frac{Q}{4\pi e_0 R} \quad \text{volts} \quad (2-158)$$

This potential is a scalar quantity.

Electric Field Intensity. The electric field intensity is the force per unit charge that would act at a point in the field on a very small test charge placed at that location. The electric field intensity E at a distance R m from a concentrated charge Q C is

$$E = \frac{Q}{4\pi e_0 R^2} \quad \text{N/C} \quad (2-159)$$

Electric Potential Gradient in Electrostatic Fields. The space rate of change of the electric potential is the electric potential gradient of the field, symbolized by $\Delta\phi$. The general relationship between the gradient of the electric potential and the electric field intensity is

$$E = -\nabla\phi \quad \text{V/m} \quad (2-160)$$

The units for the electric potential gradient, volts per meter, are frequently also used for the electric field intensity because their magnitudes are the same.

Electric Flux Density. The density of electric flux D in a region where simple dielectric materials exist is determined from the electric field intensity from

$$D = eE = e_0 K E \quad \text{C/m}^2 \quad (2-161)$$

where K is a dimensionless number called the *dielectric constant*. In free space K is unity. For numerical values of dielectric constant of various dielectrics, see Sec. 4.

Polarization. The polarization is the excess of electric flux density that results in dielectric materials over that which would result at the same electric field intensity if the space were free of material substance. Thus

$$P = D - e_0 E \quad \text{C/m}^2 \quad (2-162)$$

Crystalline Atomic Materials. In simple isotropic materials, the directions of the vectors P , D , and E are the same. For crystalline atomic structures that are not isotropic, Eq. (2-162) is the only relationship which is meaningful, and Eq. (2-161) should not be used.

Electric Flux. Electric flux and its density are related by

$$\psi = \int D \cos \alpha dA \quad \text{coulombs} \quad (2-163)$$

where α is the angle between the direction of the electric flux density D and the normal at each differential surface area dA .

Capacitance. The capacitance between two oppositely charged bodies is the ratio of the magnitude of charge on either body to the difference of electric potential between them. Thus

$$C = \frac{Q}{V} \quad \text{farads} \quad (2-164)$$

where Q is in coulombs and V is the voltage between the two equally but oppositely charged bodies, in volts.

Elastance. The reciprocal of capacitance, called *elastance*, is

$$S = V/Q \quad \text{farads} \quad (2-165)$$

Electric Field Outside an Isolated Sphere in Free Space. The electric field intensity at a distance r m from the center of an isolated charged sphere located in free space is

$$E = \frac{Q}{4\pi\epsilon_0 r^2} \quad \text{V/m} \quad (2-166)$$

where Q is the total charge (which is distributed uniformly) on the sphere.

Spherical Capacitor. The capacitance between two concentric charged spheres is

$$C = \frac{4\pi\epsilon_0 K}{1/R_1 - 1/R_2} \quad \text{farads} \quad (2-167)$$

where R_1 is the outside radius of the inner sphere, R_2 is the inside radius of the outer sphere, and K is the dielectric constant of the space between them.

Electric Field Intensity Created by an Isolated, Charged, Long Cylindrical Wire in Free Space. The electric field intensity in the vicinity of a long, charged cylinder is

$$E = \frac{\Lambda}{2\pi\epsilon_0 r} \quad \text{V/m} \quad (2-168)$$

where Λ is the charge per unit of length in coulombs per meter (distributed uniformly over the surface of the isolated cylinder) and r is the distance in meters from the center of the cylinder to the point at which the electric field intensity is evaluated.

Coaxial Cable. The capacitance per unit length of a coaxial cable composed of two concentric cylinders is

$$c = \frac{2\pi\epsilon_0 K}{\ln(R_2/R_1)} \quad \text{F/m} \quad (2-169)$$

where R_1 is the outside radius of the inner cylinder, R_2 is the inside radius of the outer cylinder, and K is the dielectric constant of the space between the cylinders.

Two-Wire Line. The capacitance per unit length between two long, oppositely charged cylindrical conductors of equal radii, parallel and external to each other, is

$$c = \frac{2\pi\epsilon_0 K}{\ln \left[\frac{D}{2R} + \sqrt{\left(\frac{D}{2R}\right)^2 - 1} \right]} \quad \text{F/m} \quad (2-170)$$

where D is the distance in meters between centers of the two cylindrical wires each with radius R and K is the uniform dielectric constant of all space external to the wires.

Capacitance of Two Flat, Parallel Conductors Separated by a Thin Dielectric. The capacitance is approximately

$$C = \frac{\epsilon_0 K A}{t} \quad \text{farads} \quad (2-171)$$

where A is the area of either of the two conductors, t is the spacing between them, and K is the dielectric constant of the space between the conductors. Strictly, the linear dimensions of the flat conductors should be very large compared with the spacing between them. Good results are obtained from Eq. (2-171) even though the conductors are curved provided that the spacing t is small with respect to the radius of curvature.

Induced Charges. The surface of a conducting body, near a charge Q , through which no currents are flowing is an equipotential surface, a condition maintained by the motion of positive and negative charges to the parts of the conductor near Q and distant from it. Hence, the potential at any point on the conductor, due to all the charges of the system, is a constant. The charges on the conductors are said to be induced by Q , and the conductor is said to be electrified by induction.

Electrostatic Induction on Parallel Wires. Two insulated wires running parallel to a wire carrying a charge A C/m display a potential difference (provided that the two wires are not connected to each other or to other conductors) of

$$\phi = \frac{\Lambda}{2\pi\epsilon_0} \ln \frac{b}{a} \quad \text{volts} \quad (2-172)$$

where b and a are the distances of the two insulated wires from the charged wire.

If the two wires are connected together, as, for example, through telephone instruments, the current flowing from one wire to the other is that required to equalize their potential difference.

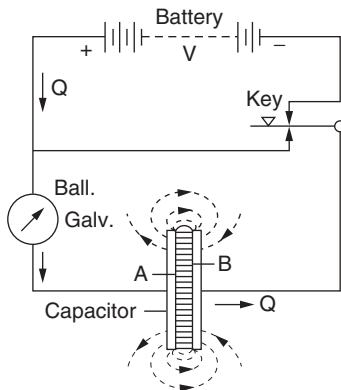


FIGURE 2-54 Circuit containing a capacitor.

2.1.27 The Dielectric Circuit

Circuit Concepts with Capacitive Elements. When a continuous voltage is applied to the terminals of a capacitor (AB , Fig. 2-54), a positive charge of electricity $+Q$ appears on one plate and a negative charge $-Q$ on the other. A quantity of electricity Q flows through the connecting wires, and this quantity of electricity is said to be displaced through the dielectric. An electrostatic field then exists

between the two charged plates. Capacitors are introduced in Sec. 2.1.7. Capacitance formulas are given in those paragraphs.

Electrostatic Flux. The space between the plates of a capacitor can be treated as a dielectric circuit through which passes a dielectric flux ψ , in coulombs. In any dielectric circuit, one coulomb of electrostatic flux passes from each coulomb of positive charge to each coulomb of negative charge, and this is true with any insulating substance or group of substances. That is, electrostatic flux lines end only on charges of electricity. Their number is not affected when they pass from one dielectric to another, unless there is a charge of electricity on the surface of separation. Electrostatic flux lines are also called *lines of electrostatic induction*.

Capacitance to Neutral of a Conductor. The capacitance to neutral of a conductor in an AC line is defined as the capacitance that, when multiplied by $2\pi f$ and by the voltage to neutral, gives the charging current of the conductor, f being the frequency. This is not the same as the capacitance to a neutral wire measured electrostatically. The voltage to neutral of a single-phase line is one-half the voltage between conductors. The voltage to neutral of a balanced 3-phase line is equal to the voltage between conductors divided by 1.732.

When the conductors are round wires, for either single-phase or 3-phase overhead lines, the capacitance to neutral is

$$C = \frac{2\pi e_0 K}{\ln \sqrt{\frac{s}{d}} + \left[\left(\frac{s}{d} \right)^2 - 1 \right]^{1/2}} \quad \text{F/m, to neutral} \quad (2-173)$$

or, approximately,

$$C = \frac{0.0388}{\ln(2s/d)} \quad \mu\text{F/mi, to neutral} \quad (2-174)$$

where s is the axial spacing and d is the diameter of the conductors, in the same units. Values of charging kVA for transmission lines are tabulated in Sec. 14.

The capacitance of a complete single-phase line is one-half the capacitance to neutral of one conductor. The capacitance of stranded conductors may be approximately calculated by using the outside diameter of the conductors. The capacitance of iron or steel conductors is calculated by the same formulas as that of copper conductors.

The preceding relations assume equilateral spacing for 3-phase systems. If unbalanced spacings are present and the phases are balanced by transposing the conductors over the length of the line, the approximate capacitance per phase can be obtained from the concepts of geometric mean distances D_m and D_s . Then, approximately,

$$C = \frac{2\pi e_0 k}{\ln(D_m D_s')} \quad \text{f/m} \quad (2-175)$$

The self-geometric mean distance n in Eq. (2-175) differs slightly from that for D_s in Eq. (2-154), in that for good conductors, the transverse gradient of the electric field is confined principally to the airspace about the conductors, and D is slightly larger than D_s of Eq. (2-154). In the latter equation, internal flux linkages in the conductor contribute to the meaning of D_s .

Velocity of Propagation on Long Transmission Lines. The inductance and capacitive parameters per unit length of a transmission line determine the velocity with which such effects as switching surges are propagated along the line. The velocity of propagation is

$$\nu = \frac{1}{\sqrt{lc}} \quad \text{m/s} \quad (2-176)$$

where l is the inductance per unit length from Eq. (2-154), and c is the capacitance per unit length from Eq. (2-175). Substituting these values gives

$$\nu = 3 \times 10^8 = \sqrt{\frac{\ln(D_m/D_s)}{K \ln(D_m/D_s)}} \quad \text{m/s} \quad (2-177)$$

The fact that D is slightly larger than D_s produces a velocity of propagation along the transmission line which is slightly less than the velocity of propagation of electromagnetic radiation in free space (3×10^8 m/s). Since the dielectric constant K of the atmosphere surrounding the transmission line may be somewhat greater than unity, the velocity of propagation may be reduced slightly more. Magnetic materials in the conductors tend to increase the inductance in the denominator of Eq. (2-176) and reduce further the velocity of propagation by a small amount.

Dielectric Strength of Insulating Materials. The dielectric strength of insulating materials (rupturing voltage gradient) is the maximum voltage per unit thickness that a dielectric can withstand in a uniform field before it breaks down electrically. The dielectric strength is usually measured in kilovolts per millimeter or per inch. It is necessary to define the dielectric strength in terms of a uniform field, for instance, between large parallel plates a short distance apart. If the striking voltage is determined between two spheres or electrodes of other defined shape, this fact must be stated. In designing insulation, a factor of safety is assumed depending upon conditions of operation. For numerical values of rupturing voltage gradients of various insulating materials, see Sec. 4.

2.1.28 Dielectric Loss and Corona

Dielectric Hysteresis and Conductance. When an alternating voltage is applied to the terminals of a capacitor, the dielectric is subjected to periodic stresses and displacements. If the material were perfectly elastic, no energy would be lost during any cycle, because the energy stored during the periods of increased voltage would be given up to the circuit when the voltage is decreased. However, since the electric elasticity of dielectrics is not perfect, the applied voltage has to overcome molecular friction or viscosity, in addition to the elastic forces. The work done against friction is converted into heat and is lost. This phenomenon resembles magnetic hysteresis (Sec. 2.1.23) in some respects but differs in others. It has commonly been called *dielectric hysteresis* but is now often called *dielectric loss*. The energy lost per cycle is proportional to the square of the applied voltage. Methods of measuring dielectric loss are described in Sec. 3.

An imperfect capacitor does not return on discharge the full amount of energy put into it. Sometime after the discharge, an additional discharge may be obtained. This phenomenon is known as *dielectric absorption*.

A capacitor that shows such a loss of power can be replaced for purposes of calculation by a perfect capacitor with an ohmic conductance shunted around it. This conductance (or "leakance") is of such value that its PR loss is equal to the loss of power from all causes in the imperfect capacitor. The actual current through the capacitor is then considered as consisting of two components—the leading reactive component through the ideal capacitor and the loss component, in phase with the voltage, through the shunted conductance.

Electrostatic Corona. When the electrostatic flux density in the air exceeds a certain value, a discharge of pale violet color appears near the adjacent metal surfaces. This discharge is called *electrostatic corona*. In the regions where the corona appears, the air is electrically ionized and is a conductor of electricity. When the voltage is raised further, a brush discharge takes place, until the whole thickness of the dielectric is broken down and a disruptive discharge, or spark, jumps from one electrode to the other.

Corona involves power loss, which may be serious in some cases, as on transmission lines (Secs. 14 and 15). Corona can form at sharp corners of high-voltage switches, bus bars, etc., so the radii of such parts are made large enough to prevent this. A voltage of 12 to 25 kV between conductors separated by a fraction of an inch, as between the winding and core of a generator or between sections of the winding of an air-blast transformer, can produce a voltage gradient sufficient to cause corona.

A voltage of 100 to 200 kV may be required to produce corona on transmission-line conductors that are separated by several feet. Corona can have an injurious effect on fibrous insulation. For numerical data in application to transmission lines see Secs. 14 and 15.

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